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Braiding structures on formal Poisson groups and classical solutions of the QYBE

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Abstract

If $\mathfrak{g}$ is a quasitriangular Lie bialgebra, the formal Poisson group $F[[\mathfrak{g}^*]]$ can be given a braiding structure. This was achieved by Weinstein and Xu using purely geometrical means, and independently by the authors by means of quantum groups. In this paper we compare these two approaches. First, we show that the braidings they produce share several similar properties (in particular, the construction is functorial); secondly, in the simplest case ($G = \mathrm{SL}_2$) they do coincide. The question then rises of whether they are always the same this is positively answered in a separate paper.

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1. Introduction

In the study of classical Hamiltonian systems, one is naturally interested in those which are completely integrable. A natural condition to achieve complete integrability for the system is that it admit a so called "Lax pair", thus one typical goal is to find Hamiltonian systems admitting such a pair; a standard recipe to obtain this has been provided by Semenov-Tian-Shansky (see [15]), which explain how to get such a system proceeding from a pair $(\mathfrak{g}, \boldsymbol{r})$ where $\mathfrak{g}$ is a Lie quasitriangular Lie bialgebra and $\boldsymbol{r}$ is its $\boldsymbol{r}$-matrix, a

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classical solution of the classical Yang–Baxter equation (CYBE). The system is built up on $\mathfrak{g}^*$, the Lie bialgebra dual to $\mathfrak{g}$, as phase space, and the $\boldsymbol{r}$-matrix $\boldsymbol{r}$ provides (a recipe for) the Poisson bracket on $C^\infty(\mathfrak{g}^*)$. This raises the question of studying quasitriangular bialgebras, as objects of special interest within the category of Lie bialgebras in particular, since we think at $\mathfrak{g}^*$ as a phase space, so that $\mathfrak{g}$ is its cotangent space, one's desire is to understand the geometrical meaning of the classical $\boldsymbol{r}$-matrix.

A second motivation for studying the geometrical meaning of the classical $\boldsymbol{r}$-matrix arises from conformal, quantum and topological quantum field theories. Indeed, all these are concerned with the notion of "fusion rules" which roughly rule the tensor product in a quasitensor category (see, e.g. [7]). As an application—among others—one has a recipe which provides tangle and link invariants as well as invariants of 3-manifolds (cf. [16]). In this setting, the common notion one start with is that of a quasitensor (or "braided monoidal") category; such an object can be built up as category of representations of a quasitriangular Hopf algebra (QTHA). Indeed, by Tannaka–Krein reconstruction theorems the two notions—quasitensor categories and QTHA—are essentially equivalent, so one may switch to the study of QTHAs. A key example of QTHA is given by a quantum group, in the shape of a quantum universal enveloping algebra (QUEA) together with its (universal) R-matrix. Now, the semiclassical counterpart of a QUEA is a Lie bialgebra $\mathfrak{g}$ (i.e. the given QUEA is the quantization of $U(\mathfrak{g})$). If the QUEA is also quasitriangular, then the semiclassical counterpart of its R-matrix is a classical $\boldsymbol{r}$-matrix $\boldsymbol{r}$ on $\mathfrak{g}$, the pair $(\mathfrak{g}, \boldsymbol{r})$ being a quasitriangular Lie bialgebra. The question then rises of whether—or at least how far—one can perform the constructions which are usually made via the QUEA and its R-matrix (such as that of link invariants) using instead only the "semiclassical" datum of $(\mathfrak{g}, \boldsymbol{r})$; then again the key point will be to understand the geometrical meaning of the classical $\boldsymbol{r}$-matrix.

With this kind of motivations, we go and study the following problem. It is known that if $\mathfrak{g}$ is a Lie bialgebra (over a field $\mathbb{K}$ of zero characteristic), then its dual space $\mathfrak{g}^*$ is a Lie bialgebra as well. Also, let G be an algebraic Poisson group—or Poisson–Lie group, say, when $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$—whose tangent Lie bialgebra is $\mathfrak{g}$. Now assume $\mathfrak{g}$ is quasitriangular, with $\boldsymbol{r}$-matrix $\boldsymbol{r}$. This gives to $\mathfrak{g}$ some additional properties; two questions then rise:

($*$) What an additional structure one obtains on the dual Lie bialgebra $\mathfrak{g}^*$?
($\bullet$) What is the geometrical global datum on G which is the result of "integrating" $\boldsymbol{r}$?

Of course, the two questions and their answers are necessarily tightly related.

First, an answer to question ($*$) was given by the authors in [13] (cf. also [8,9,14]). The topological Poisson Hopf algebra $F[[\mathfrak{g}^*]]$ (the function algebra of the formal Poisson group associated to $\mathfrak{g}^*$) is *braided* (see the definition later on).

The result in [13] was proved using the theory of quantum groups. Indeed, after Etingof–Kazhdan (cf. [5]) every Lie bialgebra admits a quantization $U_{\bar{h}}(\mathfrak{g})$, namely a (topological) Hopf algebra over $\mathbb{K}[[\bar{h}]]$ whose specialization at $\bar{h} = 0$ is isomorphic to $U(\mathfrak{g})$ as a co-Poisson Hopf algebra; in addition, if $\mathfrak{g}$ is quasitriangular and $\boldsymbol{r}$ is its $\boldsymbol{r}$-matrix, then such a $U_{\bar{h}}(\mathfrak{g})$ exists which is quasitriangular too, as a Hopf algebra, with an R-matrix $R_{\bar{h}} (\in U_{\bar{h}}(\mathfrak{g}) \otimes U_{\bar{h}}(\mathfrak{g}))$ such that $R_{\bar{h}} \equiv 1 + \boldsymbol{r}\bar{h} \bmod \bar{h}^2$ (here one identifies, as $\mathbb{K}[[\bar{h}]]$-modules, $U_{\bar{h}}(\mathfrak{g}) \cong U(\mathfrak{g})[[\bar{h}]]$). Using Drinfeld's *quantum duality principle* ([2]; cf. [12] for a proof), from any QUEA $U_{\bar{h}}(\mathfrak{g})$ with semiclassical limit $U(\mathfrak{g})$ one can extract a certain quantum formal series Hopf algebra (QFSHA) $U_{\bar{h}}(\mathfrak{g})'$ such that the semiclassical limit of $U_{\bar{h}}(\mathfrak{g})'$

is $F[[\mathfrak{g}^*]]$. In [13], starting from a quasitriangular QUEA $(U_{\hbar}(\mathfrak{g}), R)$, we showed that, although a priori $R \notin U_{\hbar}(\mathfrak{g})' \otimes U_{\hbar}(\mathfrak{g})'$ (so that the pair $(U_{\hbar}(\mathfrak{g})', R)$ is *not* in general a QTHA), nevertheless its adjoint action $\mathfrak{R}_{\hbar} := \mathrm{Ad}(R_{\hbar}) : U_{\hbar}(\mathfrak{g}) \otimes U_{\hbar}(\mathfrak{g}) \to U_{\hbar}(\mathfrak{g}) \otimes U_{\hbar}(\mathfrak{g}), x \otimes y \mapsto R_{\hbar} \cdot (x \otimes y) \cdot R_{\hbar}^{-1}$ stabilizes the subalgebra $U_{\hbar}(\mathfrak{g})' \otimes U_{\hbar}(\mathfrak{g})'$, hence induces by specialization an operator $\mathfrak{R}_0$ over $F[[\mathfrak{g}^*]] \otimes F[[\mathfrak{g}^*]]$. Moreover, the properties which make $R_{\hbar}$ an R-matrix imply that $\mathfrak{R}_{\hbar}$ is a braiding operator, hence the same holds for $\mathfrak{R}_0$. Thus, the pair $(F[[\mathfrak{g}^*]], \mathfrak{R}_0)$ is a braided Hopf algebra. In particular, this gives us a new method to produce set-theoretical solutions of the QYBE, thus giving a positive answer to a question set in [3] (also tackled, for instance, in [6]). Note also that for igniting our construction we only need a quantization functor $(\mathfrak{g}, \boldsymbol{r}) \mapsto (U_{\hbar}(\mathfrak{g}), R)$, and several of them exist (see [4]).

Secondly, an answer to question (•) was given by Weinstein and Xu in [17]. We briefly sketch their results. Let G, resp. G^*, be a Poisson group with tangent Lie bialgebra $\mathfrak{g}$, resp. $\mathfrak{g}^*$. In addition, assume both G and G^* to be complete. Let D be the corresponding double Poisson group, which is given a structure of symplectic double groupoid, over G and G^* at once (further assumptions are needed, see Section 4 later on). Then the authors prove that there is a classical analogous of the quantum R-matrix, namely a Lagrangian submanifold $\mathcal{R}$ of $D \times D$, called *the* (*global*) *classical* $\mathcal{R}$*–matrix*, which enjoys much the same properties of a quantum R-matrix! Furthermore, for any symplectic leaf S in G^*, this $\mathcal{R}$ induces a symplectic automorphism of $S \times S$ which in turn at the level of function algebras yields a braiding for $F[S]$; then, as G^* is the union of its symplectic leaves, we get also a braiding on $F[G^*]$ and so, via completion, a braiding on $F[[\mathfrak{g}^*]]$ too.

As a first goal in this paper, we investigate more in depth the properties of the construction in [17]. In particular, we show that the step $(U_{\hbar}(\mathfrak{g}), R) \mapsto (U_{\hbar}(\mathfrak{g})', \mathfrak{R}_{\hbar})$ is functorial and preserves quantization equivalence. Since the initial quantization step $(\mathfrak{g}, \boldsymbol{r}) \mapsto (U_{\hbar}(\mathfrak{g}), R_{\hbar})$ (provided by [5], but any other would work) is functorial, and of course the final specialization step $(U_{\hbar}(\mathfrak{g})', \mathfrak{R}_{\hbar}) \mapsto (F[[\mathfrak{g}^*]], \mathfrak{R}_0)$ is trivially functorial, we conclude that the whole construction $(\mathfrak{g}, \boldsymbol{r}) \mapsto (F[[\mathfrak{g}^*]], \mathfrak{R}_0)$ is functorial too. Moreover, whenever one has a braiding on $F[[\mathfrak{g}^*]]$ a so-called *infinitesimal braiding* $\bar{\mathfrak{R}}$ is defined on the cotangent Lie bialgebra of $F[[\mathfrak{g}^*]]^{\otimes 2}$, which is just $\mathfrak{g}^{\oplus 2}$. If the braiding is the afore mentioned $\mathfrak{R}_0$, we prove that the infinitesimal braiding $\bar{\mathfrak{R}}_0$ is trivial.

As a second goal of the paper, we compare our results with those of [17]. First of all, a general fact is worth stressing. The purpose in [17] is to find a geometrical counterpart of the classical $\boldsymbol{r}$-matrix, in particular an object which is of global rather than local nature: to this end, one is forced to impose some additional requirements from scratch, mainly the existence of complete Poisson groups G and G^* with tangent Lie bialgebras respectively $\mathfrak{g}$ and $\mathfrak{g}^*$. In contrast, the approach of [13] sticks to the infinitesimal level: everything is formulated in terms of Lie bialgebras or formal Poisson groups. Therefore, the final output of [17] is stronger but requires stronger hypotheses as well. Nevertheless, the additional requirements in [17] are not necessary if we stick to the infinitesimal setting: indeed, a good deal of the analysis therein can be carried out as well in local terms—just on germs of Poisson groups—so that eventually one ends up with results which are perfectly comparable with those of [13]. Thus we compare the braiding $\mathfrak{R}_{\mathrm{WX}}$ of [17] with the one of [13], call it $\mathfrak{R}_{\mathrm{GH}}$. Indeed, one has a theoretical reason to find strong similarities, namely, the construction in [13] is a *geometric* quantization of $(\mathfrak{g}, \boldsymbol{r})$, whereas the one of [13] passes through *deformation* quantization. As a matter of fact, first we show that the infinitesimal braiding $\bar{\mathfrak{R}}_{\mathrm{WX}}$ is trivial, just like $\bar{\mathfrak{R}}_{\rm GH}$. Secondly, when $\mathfrak{g} = \mathfrak{sl}_2$ with the standard $\boldsymbol{r}$-matrix we prove via explicit computation that $\bar{\mathfrak{R}}_{\rm WX} = \bar{\mathfrak{R}}_{\rm GH}$. This raises the question of whether $\bar{\mathfrak{R}}_{\rm WX}$ and $\bar{\mathfrak{R}}_{\rm GH}$ do always coincide: we give an affirmative answer in a separate paper [18].

The paper is organized as follows. Section 2 is devoted to recall some notions and results of quantum theory. Section 3 deals with the construction of braidings via quantum groups, after [13]: in particular we point out its "compatibility" with the equivalence relation for quantizations, we prove the triviality of the associated infinitesimal braiding, and we sketch some examples. Section 4 deals with the geometrical construction of braidings after [17]. In particular we reformulate some results from [loc. cit.] to make them fit with our language, and we prove that the associated infinitesimal braiding is trivial. Finally, Section 5 is devoted to explicit computation of both $\bar{\mathfrak{R}}_{\rm WX}$ and $\bar{\mathfrak{R}}_{\rm GH}$, which shows they do coincide.

2. Definitions and recalls from quantum group theory

2.1. *Topological* $\mathbb{K}[[\hbar]]$*-modules and topological Hopf* $\mathbb{K}[[\hbar]]$*-algebras*

Let $\mathbb{K}$ be a fixed field of zero characteristic, $\hbar$ an indeterminate. The ring $\mathbb{K}[[\hbar]]$ will always be considered as a topological ring w.r.t. the $\hbar$–adic topology. Let X be any $\mathbb{K}[[\hbar]]$-module. We set $X_0 := X/\hbar X = \mathbb{K} \otimes_{\mathbb{K}[[\hbar]]} X$, a $\mathbb{K}$-module (via scalar restriction $\mathbb{K}[[\hbar]] \to \mathbb{K}[[\hbar]]/\hbar\mathbb{K}[[\hbar]] \cong \mathbb{K}$) which we call the *specialization* of X at $\hbar = 0$, or *semiclassical limit* of X; we shall also use notation $X \overset{\hbar\to 0}{\longrightarrow} \bar{Y}$ to mean $X_0 \cong \bar{Y}$. For later use, we also set ${}^F X := \mathbb{K}((h)) \otimes_{\mathbb{K}[[\hbar]]} X$, a vector space over $\mathbb{K}((h))$.

Let $\mathcal{T}_{\hat{\otimes}}$ be the category whose objects are all topological $\mathbb{K}[[\hbar]]$-modules which are topologically free (i.e.isomorphic to $V[[\hbar]]$ for some $\mathbb{K}$–vector space V, with the $\hbar$–adic topology) and whose morphisms are the $\mathbb{K}[[\hbar]]$-linear maps (which are automatically continuous). This is a tensor category w.r.t. the tensor product $T_1 \hat{\otimes} T_2$ defined to be the separated $\hbar$–adic completion of the algebraic tensor product $T_1 \otimes_{\mathbb{K}[[\hbar]]} T_2$ (for all $T_1, T_2 \in \mathcal{T}_{\hat{\otimes}}$).

Let $\mathcal{P}_{\tilde{\otimes}}$ be the category whose objects are all topological $\mathbb{K}[[\hbar]]$-modules isomorphic to modules of the type $\mathbb{K}[[\hbar]]^E$ (the Cartesian product indexed by E, with the Tikhonov product topology) for some set E. These are complete w.r.t. the weak topology, in fact they are isomorphic to the projective limit of their finite free submodules (each one taken with the $\hbar$–adic topology); the morphisms in $\mathcal{P}_{\tilde{\otimes}}$ are the $\mathbb{K}[[\hbar]]$-linear continuous maps. This is a tensor category w.r.t. the tensor product $P_1 \tilde{\otimes} P_2$ defined to be the completion of the algebraic tensor product $P_1 \otimes_{\mathbb{K}[[\hbar]]} P_2$ w.r.t. the weak topology. Therefore $P_i \cong \mathbb{K}[[\hbar]]^{E_i}$ $(i = 1, 2)$ yields $P_1 \tilde{\otimes} P_2 \cong \mathbb{K}[[\hbar]]^{E_1 \times E_2}$ (for all $P_1, P_2 \in \mathcal{P}_{\tilde{\otimes}}$).

Note that the objects of $\mathcal{T}_{\hat{\otimes}}$ and of $\mathcal{P}_{\tilde{\otimes}}$ are complete and separated w.r.t. the $\hbar$–adic topology, whence one has $X \cong X_0[[\hbar]]$ (as $\mathbb{K}[[\hbar]]$-modules) for each of them.

To simplify notation, in the sequel we shall usually write simply $\otimes$ for either $\hat{\otimes}$ or $\tilde{\otimes}$.

Definition 2.1 (cf. [2,3], Section 7).

(a) We call quantized universal enveloping algebra (in short, QUEA) any Hopf algebra H in the category $\mathcal{T}_{\hat{\otimes}}$ such that $H_0 := H/\hbar H$ is a co-Poisson Hopf algebra isomorphic to $U(\mathfrak{g})$ for some finite-dimensional Lie bialgebra $\mathfrak{g}$ (over $\mathbb{K}$); in this case we write

$H = U_{\overline{h}}(\mathfrak{g})$, and say H is a *quantization* of $U(\mathfrak{g})$. We call $\mathcal{QUEA}$ the subcategory of $\mathcal{T}_{\hat{\otimes}}$ whose objects are QUEA (relative to all possible $\mathfrak{g}$), with the obvious morphisms.

(b) We call quantized formal series Hopf algebra (in short, QFSHA) any Hopf algebra K in the category $\mathcal{P}_{\tilde{\otimes}}$ such that $K_0 := K/\overline{h}K$ is a topological Poisson Hopf algebra isomorphic to $F[[\mathfrak{g}]]$ for some finite-dimensional Lie bialgebra $\mathfrak{g}$ (over $\mathbb{K}$); then we write $H = F_{\overline{h}}[[\mathfrak{g}]]$, and say K is a *quantization* of $F[[\mathfrak{g}]]$. We call $\mathcal{QFSHA}$ the full subcategory of $\mathcal{P}_{\tilde{\otimes}}$ whose objects are QFSHA (relative to all possible $\mathfrak{g}$), with the obvious morphisms.

(c) If H_1, H_2, are two quantizations of $U(\mathfrak{g})$, resp. of $F[[\mathfrak{g}]]$ (for some Lie bialgebra $\mathfrak{g}$), we say that H_1 is equivalent to H_2, and we write $H_1 \equiv H_2$, if there is an isomorphism $\varphi : H_1 \cong H_2$ (in $\mathcal{QUEA}$, resp. in $\mathcal{QFSHA}$) and a $\mathbb{K}[[\overline{h}]]$-linear isomorphism $\varphi_+ : H_1 \cong H_2$ such that $\varphi = \mathrm{id} + \overline{h}\varphi_+$ when identifying H_1 and H_2 with $U(\mathfrak{g})[[\overline{h}]]$, resp. with $F[[\mathfrak{g}]][[\overline{h}]]$.

Remark 2.2.

(a) Note that the objects of $\mathcal{QUEA}$ and of $\mathcal{QFSHA}$ are *topological* Hopf algebras, not standard ones. As a matter of notation, if H is any Hopf algebra (maybe topological), we shall denote by m its product, by 1 its unit element, by Δ its coproduct, by ϵ its counit and by S its antipode (with a subscript H if necessary).

(b) If $H \in \mathcal{HA}_{\hat{\otimes}}$ is such that $H_0 := H/\overline{h}H$ as a Hopf algebra is isomorphic to $U(\mathfrak{g})$ for some Lie algebra $\mathfrak{g}$, then $H_0 = U(\mathfrak{g})$ is also a *co-Poisson* Hopf algebra, w.r.t. the Poisson cobracket δ defined as follows. If $x \in H_0$ and $x' \in H$ gives $x = x' + hH$, then $\delta(x) := (h^{-1}(\Delta(x') - \Delta^{\mathrm{op}}(x'))) + hH\hat{\otimes}H$; then (by [2,3], Section 3, Theorem 2) the restriction of δ makes $\mathfrak{g}$ into a Lie bialgebra. Similarly, if $K \in \mathcal{HA}_{\tilde{\otimes}}$ is such that $K_0 := K/\overline{h}K$ is a topological Poisson Hopf algebra isomorphic to $F[[\mathfrak{g}]]$ for some Lie algebra $\mathfrak{g}$ then $K_0 = F[[\mathfrak{g}]]$ is also a topological *Poisson* Hopf algebra, w.r.t. the Poisson bracket $\{,\}$ defined as follows. If $x, y \in K_0$ and $x', y' \in K$ give $x = x' + hK$, $y = y' + hK$, then $\{x, y\} := (h^{-1}(x'y' - y'x')) + hK$; then $\mathfrak{g}$ is a bialgebra again. These natural co-Poisson and Poisson structures are the ones considered in Definition 2.1.

(c) Clearly $\mathcal{QUEA}$, resp. $\mathcal{QFSHA}$, is a *tensor* subcategory of $\mathcal{T}_{\hat{\otimes}}$, resp. of $\mathcal{P}_{\tilde{\otimes}}$.

(d) We make a finiteness assumption on $\dim(\mathfrak{g})$, but infinite-dimensional cases can also be "reasonably" handled as explained in [12, Section 3.9].

2.2. *Drinfeld's functors*

Let H be a Hopf algebra (of any type) over $\mathbb{K}[[\overline{h}]]$. For each $n \in \mathbb{N}$, define $\Delta^n : H \to H^{\otimes n}$ by $\Delta^0 := \epsilon$, $\Delta^1 := \mathrm{id}_H$, and $\Delta^n := (\Delta \otimes \mathrm{id}_H^{\otimes(n-2)}) \circ \Delta^{n-1}$ if $n \geq 2$. Then set $\delta_n = (\mathrm{id}_H - \epsilon)^{\otimes n} \circ \Delta^n$, for all $n \in \mathbb{N}_+$. Finally, define

$$H' := \{a \in H | \delta_n(a) \in \overline{h}^n H^{\otimes n} \,\forall\, n \in \mathbb{N}\} \quad (\subseteq H).$$

Now let $I_H := \epsilon_H^{-1}(\overline{h}\mathbb{K}[[\overline{h}]])$; set $H^\times := \sum_{n\geq 0} \overline{h}^{-n} I_H^n = \cup_{n\geq 0}(\overline{h}^{-1}I_H)^n$ $(\subseteq {}^F H)$, and $H^\vee :=$ (separated)$\overline{h}$-adic completion of the $\mathbb{K}[[\overline{h}]]$-module $H^\times$.

The following is the first important result we need the following theorem.

Theorem 2.3 (The quantum duality principle; cf. [12, Theorem 2.3]). *The assignments $H \mapsto H^\vee$ and $H \mapsto H'$, respectively, define functors of tensor categories $\mathcal{QFSHA} \to \mathcal{QUEA}$ and $\mathcal{QUEA} \to \mathcal{QFSHA}$. These functors are inverse to each other. Indeed, for all $U_\hbar(\mathfrak{g}) \in \mathcal{QUEA}$ and all $F_\hbar[[\mathfrak{g}]] \in \mathcal{QFSHA}$ one has*

$$\frac{U_\hbar(\mathfrak{g})'}{\hbar U_\hbar(\mathfrak{g})'} = F[[\mathfrak{g}^*]], \qquad \frac{F_\hbar[[\mathfrak{g}]]^\vee}{\hbar F_\hbar[[\mathfrak{g}]]^\vee} = U(\mathfrak{g}^*)$$

(*where $\mathfrak{g}^*$ is the dual to $\mathfrak{g}$*), *i.e. $U_\hbar(\mathfrak{g})' = F_\hbar[[\mathfrak{g}^*]]$ and $F_\hbar[[\mathfrak{g}]]^\vee = U_\hbar(\mathfrak{g}^*)$. Moreover, the functors preserve equivalence, i.e. $H_1 \equiv H_2$ implies $H_1^\vee \equiv H_2^\vee$ or $H_1' \equiv H_2'$.*

2.3. An explicit description of $U_\hbar(\mathfrak{g})'$

Given any QUEA, say $U_\hbar(\mathfrak{g})$, we can give a rather explicit description of $U_\hbar(\mathfrak{g})'$. In fact, one has the following (see [12, Section 3.5]).

Given a basis $\{\bar{x}_1, \dots, \bar{x}_d\}$ of $\mathfrak{g}$, there is a lift $\{x_1, \dots, x_d\}$ of it in $U_\hbar(\mathfrak{g})$ such that $\epsilon(x_i) = 0$ and $U_\hbar(\mathfrak{g})'$ is just the topological $\mathbb{K}[[\hbar]]$-algebra in $\mathcal{P}_{\tilde{\otimes}}$ generated (*topologically*) *by $\{\hbar x_1, \dots, \hbar x_d\}$; so $U_\hbar(\mathfrak{g})' = \{\sum_{\underline{e}\in\mathbb{N}^d} a_{\underline{e}} \hbar^{|\underline{e}|} x^{\underline{e}} | a_{\underline{e}} \in \mathbb{K}[[\hbar]]\, \forall\, \underline{e}\}$ as a subset of $U_\hbar(\mathfrak{g})$.*

Hereafter, we use notation $x^{\underline{e}} := \prod_{i=1}^d x_i^{e_i}$ and $|\underline{e}| := \sum_{i=1}^d e_i$ for all $\underline{e} = (e_1, \dots, e_d) \in \mathbb{N}^d$.

Definition 2.4 (cf. [1,2,14]).

(a) A Hopf algebra H (in any tensor category) is called *quasitriangular* if there is $R \in H \otimes H$ (tensor product within the category), called *the R-matrix of H*, such that

$$R \cdot \Delta(a) \cdot R^{-1} = \mathrm{Ad}(R)(\Delta(a)) = \Delta^{\mathrm{op}}(a), \qquad (\Delta \otimes \mathrm{id})(R) = R_{13}R_{23},$$
$$(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}, \tag{2.1}$$

where $\Delta^{\mathrm{op}} := \sigma \circ \Delta(a)$ with $\sigma : H^{\otimes 2} \to H^{\otimes 2}$, $a \otimes b \mapsto b \otimes a$, and $R_{12}, R_{13}, R_{23} \in H^{\otimes 3}$, $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$, $R_{13} = (\sigma \otimes \mathrm{id})(R_{23}) = (\mathrm{id} \otimes \sigma)(R_{12})$. The algebra is called *triangular*, and the R-matrix *unitary*, if in addition $R^{-1} = R^{\mathrm{op}} := \sigma(R)$.

We call $\mathcal{QTQUEA}$, resp. $\mathcal{TQUEA}$, the subcategory of $\mathcal{QUEA}$ whose objects are all the quasitriangular, resp. the triangular, QUEA (in short QTQUEA, resp. TQUEA) and whose morphisms $\varphi : H_1 \to H_2$ enjoy $\phi^{\otimes 2}(R_1) = R_2$.

(b) A Hopf algebra H (in any tensor category) is called *braided* if there is an *algebra automorphism* $\mathfrak{R} : H \otimes H \to H \otimes H$ in the category, called *the braiding operator* (*or simply the braiding*) *of H*, different from $\sigma : a \otimes b \mapsto b \otimes a$ and such that

$$\mathfrak{R} \circ \Delta = \Delta^{\mathrm{op}}, \qquad (\Delta \otimes \mathrm{id}) \circ \mathfrak{R} = \mathfrak{R}_{13} \circ \mathfrak{R}_{23} \circ (\Delta \otimes \mathrm{id}),$$
$$(\mathrm{id} \otimes \Delta) \circ \mathfrak{R} = \mathfrak{R}_{13} \circ \mathfrak{R}_{12} \circ (\mathrm{id} \otimes \Delta), \tag{2.2}$$

where $\mathfrak{R}_{12}, \mathfrak{R}_{13}, \mathfrak{R}_{23}$ are the automorphisms of $H^{\otimes 3}$ defined by $Re_{12} = \mathfrak{R} \otimes \mathrm{id}$, $\mathfrak{R}_{23} = \mathrm{id} \otimes \mathfrak{R}$, $\mathfrak{R}_{13} = (\sigma \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathfrak{R}) \circ (\sigma \otimes \mathrm{id})$. Moreover, the braiding operator is said to be *unitary* and the algebra to be *rigid* if in addition $\mathfrak{R}^{-1} = \sigma \circ \mathfrak{R} \circ \sigma$.

We call $\mathcal{BQFSHA}$, resp. $\mathcal{RBQFSHA}$, the subcategory of $\mathcal{QFSHA}$ whose objects are all the braided, resp. the rigid braided, QFSHA (in short BQFSHA, resp. RBQFSHA) and whose morphisms $\psi : H_1 \to H_2$ enjoy $\psi^{\otimes 2} \circ \mathfrak{R}_1 = \mathfrak{R}_2 \circ \psi^{\otimes 2}$.

(c) Let $(H_1, R_1), (H_2, R_2) \in \mathcal{QTQUEA}$. We say that (H_1, R_1) is equivalent to (H_2, R_2), and we write $(H_1, R_1) \equiv (H_2, R_2)$, if $H_1 \equiv H_2$ in $\mathcal{QUEA}$ via an equivalence $\varphi : H_1 \cong H_2$ which is also an isomorphism in $\mathcal{QTQUEA}$ (i.e. such that $\phi^{\otimes 2}(R_1) = R_2$).

(d) Let $(H_1, \mathfrak{R}_1), (H_2, \mathfrak{R}_2) \in \mathcal{BQFSHA}$. We say that $(H_1, \mathfrak{R}_1)$ *is equivalent to* $(H_2, \mathfrak{R}_2)$, and we write $(H_1, \mathfrak{R}_1) \equiv (H_2, \mathfrak{R}_2)$, if $H_1 \equiv H_2$ in $\mathcal{QUEA}$ via an equivalence which is also an isomorphism in $\mathcal{BQFSHA}$ (i.e. such that $\psi^{\otimes 2} \circ \mathfrak{R}_1 = \mathfrak{R}_2 \circ \psi^{\otimes 2}$).

Remark 2.5.

(a) It follows immediately from (2.1) that R *is a solution of the quantum Yang–Baxter equation* (in short, QYBE) *in* $H^{\otimes 3}$, namely $R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$. This is the starting point for defining a braid group action on the tensor products of H-modules, and then for constructing link invariants, following [16] (see also [1, Section 15]).

Similarly, it follows from (2.2) that $\mathfrak{R}$ *is a solution of the QYBE in End*$(H^{\otimes 3})$, namely $\mathfrak{R}_{12} \circ \mathfrak{R}_{13} \circ \mathfrak{R}_{23} = \mathfrak{R}_{23} \circ \mathfrak{R}_{13} \circ \mathfrak{R}_{12}$. Again, this implies the existence of a braid group action on the tensor powers of H, from which one can start a search for link invariants.

(b) It is proved in [5] that, for any Lie bialgebra $\mathfrak{g}$, there exists a QUEA, which we'll denote $U_{\hbar}(\mathfrak{g})$, whose semiclassical limit is isomorphic to $U(\mathfrak{g})$; moreover, one has an identification $U_{\hbar}(\mathfrak{g}) \cong U(\mathfrak{g})[[\hbar]]$ as $\mathbb{K}[[\hbar]]$-modules, hence also $U_{\hbar}(\mathfrak{g}) \otimes U_{\hbar}(\mathfrak{g}) \cong (U(\mathfrak{g}) \otimes U(\mathfrak{g}))[[\hbar]]$. Here, like elsewhere in the following, the tensor products among $\mathbb{K}[[\hbar]]$-modules are topological tensor products. In addition, if $\mathfrak{g}$ is quasitriangular—as a Lie bialgebra (cf. [1])—and $\boldsymbol{r}$ is its $\boldsymbol{r}$-matrix, then there exists such a $U_{\hbar}(\mathfrak{g})$ which is quasitriangular as well—as a Hopf algebra—with an R-matrix $R_{\hbar} (\in U_{\hbar}(\mathfrak{g}) \otimes U_{\hbar}(\mathfrak{g}))$ such that $R_{\hbar} \equiv 1 + \hbar \boldsymbol{r} | \hbar^2$, that is to say $R_{\hbar} = 1 + \hbar \boldsymbol{r} + \mathcal{O}(\hbar^2)$ with $\mathcal{O}(\hbar^2) \in \hbar^2 \cdot U_{\hbar}(\mathfrak{g}) \otimes U_{\hbar}(\mathfrak{g})$.

3. Braidings from deformation quantization

Theorem 3.1 ([14, Théorème 2.1]). *Let H be a QTQUEA, and let R be its R-matrix. Then the inner automorphism* $\mathrm{Ad}(R) : H \otimes H \to H \otimes H$ *of $H \otimes H$ restricts to an automorphism of $H' \otimes H'$, and the pair* $(H', \mathrm{Ad}(R)|_{H' \otimes H'})$ *is a BQFSHA.*

As a first goal in this section we provide some further details about Theorem 3.1.

Theorem 3.2.

(a) *The functor* $(\,)' : \mathcal{QUEA} \to \mathcal{QFSHA}$ *yields by restriction two functors*

$$(\,)' : \mathcal{QTQUEA} \to \mathcal{BQFSHA}, \qquad (H, R) \mapsto (H', \mathrm{Ad}(R)|_{H' \otimes H'}),$$

$$(\,)' : \mathcal{TQUEA} \to \mathcal{RBQFSHA}, \qquad (H, R) \mapsto (H', \mathrm{Ad}(R)|_{H' \otimes H'}.$$

(b) *The functors in* (a) *preserve equivalence classes, i.e. if* $(H_1, R_1) \equiv (H_2, R_2)$ *in* $\mathcal{QTQUEA}$ *then* $(H_1', \mathrm{Ad}(R)|_{H_1'\otimes H_1'}) \equiv (H_2', \mathrm{Ad}(R)|_{H_2'\otimes H_2'})$ in $\mathcal{BQFSHA}$.

Proof.

(a) Theorem 3.1 tells that the functor $(\,)' : \mathcal{QTQUEA} \to \mathcal{BQFSHA}$ is well-defined on objects. Moreover, if $\phi : (H_1, R_1) \to (H_2, R_2)$ is a morphism in $\mathcal{QTQUEA}$ then $\phi^{\otimes 2}(R_1) = R_2$, whence $\phi^{\otimes 2} \circ \mathrm{Ad}(R_1)|_{(H_1')^{\otimes 2}} = \mathrm{Ad}(R_2)|_{(H_2')^{\otimes 2}} \circ \phi^{\otimes 2}$ follows at once, hence $\phi' := \phi|_{H_1'} : H_1' \to H_2'$ is a morphism in $\mathcal{BQFSHA}$. In addition, if $(H, R) \in \mathcal{TQUEA}$ then $R^{-1} = \sigma(R)$ yields $(\mathrm{Ad}(R)|_{(H')^{\otimes 2}})^{-1} = \mathrm{Ad}(R^{-1})|_{(H')^{\otimes 2}} = \mathrm{Ad}(\sigma(R))|_{(H')^{\otimes 2}} = \sigma \circ \mathrm{Ad}(\sigma(R))|_{(H')^{\otimes 2}} \circ \sigma$, hence $\mathrm{Ad}(\sigma(R))|_{(H')^{\otimes 2}}$ is unitary, q.e.d.

(b) This follows easily from (a) and the very definitions. □

Secondly, as a consequence of Theorem 3.1 along with the existence of quasitriangular quantization of any quasitriangular Lie bialgebra (cf. [5]) one gets a braiding on $F[[\mathfrak{g}^*]]$.

Corollary 3.3 ([14, Théorème 3.2]). *Let* $\mathfrak{g}$ *be a* (*finite-dimensional*) *quasitriangular Lie bialgebra. Then the topological Poisson Hopf algebra* $F[[\mathfrak{g}^*]]$ *is braided* (*in particular, its braiding is a Poisson automorphism*). *Moreover, there is a quantization of* $F[[\mathfrak{g}^*]]$ *which is a braided Hopf algebra whose braiding operator specializes into that of* $F[[\mathfrak{g}^*]]$.

3.1. *The triviality of the infinitesimal braiding*

Let $\mathfrak{g}$ and $\mathfrak{g}^*$ be finite-dimensional Lie bialgebras dual to each other. Assume $F[[\mathfrak{g}^*]]$ is braided (as a Poisson Hopf algebra), $\mathfrak{R}$ being its braiding (which is a Poisson automorphism also). Let $\mathfrak{m}_e^{\otimes}$ be the (unique) maximal ideal of $F[[\mathfrak{g}^* \oplus \mathfrak{g}^*]] = F[[\mathfrak{g}^*]] \otimes F[[\mathfrak{g}^*]]$ (topological tensor product, after [2,3], Chapter 1). Since $\mathfrak{R}$ is an algebra automorphism, $\mathfrak{R}(\mathfrak{m}_e^{\otimes}) = \mathfrak{m}_e^{\otimes}$ and $\mathfrak{R}$ induces an automorphism $\bar{\mathfrak{R}}$ of the vector space $\mathfrak{m}_e^{\otimes}/(\mathfrak{m}_e^{\otimes})^2$. Now, $\mathfrak{m}_e^{\otimes}/(\mathfrak{m}_e^{\otimes})^2$ with the Lie bracket induced by the Poisson bracket of $F[[\mathfrak{g}^* \oplus \mathfrak{g}^*]]$ identifies with the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$; since $\mathfrak{R}$ is also an automorphism of Poisson algebras, the map $\bar{\mathfrak{R}}$ is an automorphism of the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$; of course $\bar{\mathfrak{R}}$ inherits also other properties of the braiding $\mathfrak{R}$, in particular $\mathfrak{R}$ and $\bar{\mathfrak{R}}$ are solutions of the QYBE, hence we call it *the infinitesimal braiding* associated to $\mathfrak{R}$.

Now assume in addition that $\mathfrak{g}$ be quasitriangular, and the braiding $\mathfrak{R}$ on $F[[\mathfrak{g}^*]]$ is provided as in Corollary 3.3. Namely, let $(U_{\bar{h}}(\mathfrak{g}), R) \in \mathcal{QTQUEA}$ be a quantization of the quasitriangular Lie bialgebra $(\mathfrak{g}, \boldsymbol{r})$: by definition, this means that $U_{\bar{h}}(\mathfrak{g})$ has semiclassical limit (i.e. specialization at $\bar{h} = 0$) the co-Poisson Hopf algebra $U(\mathfrak{g})$ and, in the identification $U_{\bar{h}}(\mathfrak{g}) = U(\mathfrak{g})[[\bar{h}]]$ (as topological $\mathbb{K}[[\bar{h}]]$-modules), $R = 1 + \bar{h}\boldsymbol{r} + \mathcal{O}(\bar{h}^2)$ for some $\mathcal{O}(\bar{h}^2) \in \bar{h}^2 U(\mathfrak{g})[[\bar{h}]]$. Then $\mathfrak{R}$ is the braiding of $F[[\mathfrak{g}^* \oplus \mathfrak{g}^*]] = U_{\bar{h}}(\mathfrak{g})' \otimes U_{\bar{h}}(\mathfrak{g})' \bmod \bar{h}$ which is obtained as specialization at $\bar{h} = 0$ of $\mathrm{Ad}(R)|_{U_{\bar{h}}(\mathfrak{g})' \otimes U_{\bar{h}}(\mathfrak{g})'}$, thanks to Theorem 3.1. Then our next result is that the associated infinitesimal braiding $\bar{\mathfrak{R}}$ is always trivial.

Theorem 3.4. *The infinitesimal braiding* $\bar{\mathfrak{R}} : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$ *is trivial, i.e.* $\bar{\mathfrak{R}} = \mathrm{id}_{\mathfrak{g}\oplus\mathfrak{g}}$.

Proof. Let $\{\bar{x}_1, \ldots, \bar{x}_d\}$ be a basis of $\mathfrak{g}$, and pick a lift $\{x_1, \ldots, x_d\}$ of it in $U_{\bar{h}}(\mathfrak{g})$ as explained in Section 2.3, so that $U_{\bar{h}}(\mathfrak{g})' = \{\sum_{\underline{e}\in\mathbb{N}^d} a_{\underline{e}}\bar{h}^{|\underline{e}|}x^{\underline{e}} = \sum_{\underline{e}\in\mathbb{N}^d} a_{\underline{e}}\tilde{x}^{\underline{e}} | a_{\underline{e}} \in \mathbb{K}[[\bar{h}]]\,\forall\,\underline{e}\}$, where $\tilde{x}_i := \bar{h}x_i$ (for all i) are topological generators of $U_{\bar{h}}(\mathfrak{g})'$. Then $U_{\bar{h}}(\mathfrak{g})' \otimes U_{\bar{h}}(\mathfrak{g})'$ is generated by the ${}_1\tilde{x}_i := \tilde{x}_i \otimes 1$ and the ${}_2\tilde{x}_i := 1 \otimes \tilde{x}_i$, for all i. On the other hand, one has $U_{\bar{h}}(\mathfrak{g}) = \mathbb{K}[x_1, \ldots, x_d][[\bar{h}]]$ as topological $\mathbb{K}[[\bar{h}]]$-modules, whence $U_{\bar{h}}(\mathfrak{g}) \otimes U_{\bar{h}}(\mathfrak{g}) = (\mathbb{K}[{}_1x_1, \ldots, {}_1x_d, {}_2x_1, \ldots, {}_2x_d])[[\bar{h}]]$. Then we have an $\bar{h}$–adic expansion of R and of R^{-1}, namely $R = \sum_{n\geq 0} P_n^+({}_1\underline{x}; {}_2\underline{x})\bar{h}^n$, $R^{-1} = \sum_{m\geq 0} P_m^-({}_1\underline{x}; {}_2\underline{x})\bar{h}^m$ for some polynomials $P_n^+({}_1\underline{x}; {}_2\underline{x}) = P_n^+({}_1x_1, \ldots, {}_1x_d; {}_2x_1, \ldots, {}_2x_d)$, $P_m^-({}_1\underline{x}{}_2\underline{x}) = P_m^-({}_1x_1, \ldots, {}_1x_d; {}_2x_1, \ldots, {}_2x_d)$. Now, the condition $R = 1^{\otimes} + \bar{h}\boldsymbol{r} + \mathcal{O}(\bar{h}^2)$ (with $1^{\otimes} := 1 \otimes 1$) forces $P_0^+ = 1 = P_0^-$, $P_1^+ = \sum_{i,j} c_{i,j} \cdot {}_1x_i{}_2x_j = -P_1^+$ for some $c_{i,j} \in \mathbb{K}$ such that $\boldsymbol{r} = \sum_{i,j} c_{i,j} \cdot \bar{x}_i \otimes \bar{x}_j$. In addition, any R-matrix enjoys $(\epsilon \otimes \mathrm{id})(R) = 1 = (\mathrm{id} \otimes \epsilon)(R)$, hence also $(\epsilon \otimes \mathrm{id})(R^{-1}) = 1 = (\mathrm{id} \otimes \epsilon)(R^{-1})$; setting $P_{\pm} := R^{\pm 1} - 1$, this implies $(\epsilon \otimes \mathrm{id})(P_{\pm}) = 0 = (\mathrm{id} \otimes \epsilon)(P_{\pm})$.

Now, for all ℓ consider $(\mathrm{Ad}(R))({}_1\tilde{x}_\ell) = R \cdot {}_1\tilde{x}_\ell \cdot R^{-1}$: we have

$$\begin{aligned}(\mathrm{Ad}(R))({}_1\tilde{x}_\ell) &= R \cdot {}_1\tilde{x}_\ell \cdot R^{-1} = (1 + P_+) \cdot {}_1\tilde{x}_\ell \cdot (1 + P_-) \\ &= {}_1\tilde{x}_\ell + P_+ \cdot {}_1\tilde{x}_\ell + {}_1\tilde{x}_\ell \cdot P_- + P_+ \cdot {}_1\tilde{x}_\ell \cdot P_-. \end{aligned} \tag{3.1}$$

We know that this element belongs to $(U_{\bar{h}}(\mathfrak{g}) \otimes U_{\bar{h}}(\mathfrak{g}))' = \mathbb{K}[[{}_1\tilde{x}_1, \ldots, {}_1\tilde{x}_d, {}_2\tilde{x}_1, \ldots, {}_2\tilde{x}_d, h]]$, so we can write it as a series; since $(\epsilon \otimes \mathrm{id})(P_{\pm}) = 0$ and $(\epsilon \otimes \mathrm{id})$ is a morphism we have $(\epsilon \otimes \mathrm{id})(P_+ \cdot {}_1\tilde{x}_\ell + {}_1\tilde{x}_\ell \cdot P_- + P_+ \cdot {}_1\tilde{x}_\ell \cdot P_-) = 0$. Recalling that $\epsilon({}_1\tilde{x}_\ell) = 0$ this means that

$$P_+ \cdot {}_1\tilde{x}_\ell + {}_1\tilde{x}_\ell \cdot P_- + P_+ \cdot {}_1\tilde{x}_\ell \cdot P_- = \sum_{\underline{e}^{(1)}, \underline{e}^{(2)} \in \mathbb{N}^d} a_{\underline{e}^{(1)}, \underline{e}^{(2)}}\, {}_1\tilde{x}^{\underline{e}^{(1)}}\, {}_2\tilde{x}^{\underline{e}^{(2)}}$$

(where $a_{\underline{e}^{(1)}, \underline{e}^{(2)}} \in \mathbb{K}[[\bar{h}]]$ for all $\underline{e}^{(1)}, \underline{e}^{(2)}$) with $a_{\underline{e}^{(1)}, \underline{0}} = 0 = a_{\underline{0}, \underline{e}^{(2)}}$ for all $\underline{e}^{(1)}, \underline{e}^{(2)}$, thus

$$P_+ \cdot {}_1\tilde{x}_\ell + {}_1\tilde{x}_\ell \cdot P_- + P_+ \cdot {}_1\tilde{x}_\ell \cdot P_- = \sum_{|\underline{e}^{(1)}|, |\underline{e}^{(2)}| > 1} a_{\underline{e}^{(1)}, \underline{e}^{(2)}}\, {}_1\tilde{x}^{\underline{e}^{(1)}}\, {}_2\tilde{x}^{\underline{e}^{(2)}}. \tag{3.2}$$

Now ${}_1\tilde{x}^{\underline{e}^{(1)}}, {}_2\tilde{x}^{\underline{e}^{(2)}} \bmod \bar{h}(U_{\bar{h}}(\mathfrak{g})^{\otimes 2})'$ belong to $\mathfrak{m}_e^{\otimes}$ (notation of Section 3.1) as soon as $|\underline{e}^{(1)}| > 1$, $|\underline{e}^{(2)}| > 1$; so (3.2) gives

$$(P_+ \cdot {}_1\tilde{x}_\ell + {}_1\tilde{x}_\ell \cdot P_- + P_+ \cdot {}_1\tilde{x}_\ell \cdot P_-) \equiv 0 \bmod \bar{h}\,(U_{\bar{h}}(\mathfrak{g})^{\otimes 2})' \in (\mathfrak{m}_e^{\otimes})^2$$

and this along with (3.1) yields (for all $\ell = 1, \ldots, d$)

$$\begin{aligned}&\mathfrak{R}(({}_1\tilde{x}_\ell \bmod \bar{h}(U_{\bar{h}}(\mathfrak{g})^{\otimes 2})') \bmod (\mathfrak{m}_e^{\otimes})^2) \\ &\quad = ((\mathrm{Ad}(R))({}_1\tilde{x}_\ell) \bmod \bar{h}(U_{\bar{h}}(\mathfrak{g})^{\otimes 2})') \bmod (\mathfrak{m}_e^{\otimes})^2 \\ &\quad = ({}_1\tilde{x}_\ell \bmod \bar{h}(U_{\bar{h}}(\mathfrak{g})^{\otimes 2})') \bmod (\mathfrak{m}_e^{\otimes})^2.\end{aligned}$$

Similarly one gets (for all $\ell = 1, \ldots, d$)

$$\mathfrak{R}(({}_2\tilde{x}_\ell \bmod \bar{h}(U_{\bar{h}}(\mathfrak{g})^{\otimes 2})') \bmod (\mathfrak{m}_e^{\otimes})^2) = ({}_2\tilde{x}_\ell \bmod \bar{h}(U_{\bar{h}}(\mathfrak{g})^{\otimes 2})') \bmod (\mathfrak{m}_e^{\otimes})^2.$$

Letting ${}_s\check{x}_\ell := ({}_s\tilde{x}_\ell \bmod \bar{h}(U_{\bar{h}}(\mathfrak{g})^{\otimes 2})') \bmod (\mathfrak{m}_e^{\otimes})^2 \in \mathfrak{m}_e^{\otimes}/(\mathfrak{m}_e^{\otimes})^2 = \mathfrak{g} \oplus \mathfrak{g}$ (for all $s = 1, 2$ and $\ell = 1, \ldots, d$), we have in short $\mathfrak{R}({}_s\check{x}_\ell) = {}_s\check{x}_\ell$ for all s, ℓ. Since the ${}_s\tilde{x}_\ell$ generate $(U_{\bar{h}}(\mathfrak{g})^{\otimes 2})'$, the ${}_s\check{x}_\ell$ span $\mathfrak{g} \oplus \mathfrak{g}$, hence we can conclude that $\mathfrak{R}$ is trivial, as claimed. □

3.2. The example of semisimple and (untwisted) affine cases

In [8,9,14] the adjoint action of the R-matrix of the Jimbo–Lusztig's quantum groups $U_q(\mathfrak{g})$ was studied. In this section we briefly outline how the results therein can be read as special occurrences of the ones cited here, namely the existence of braidings on $U_{\bar{h}}(\mathfrak{g})$.

Let $\mathfrak{g} = \mathfrak{g}^\tau$ be a semisimple Lie algebra, i.e. a finite type Kac–Moody algebra, endowed with the Lie cobracket—depending on the parameter τ—given in [10, Section 1.3], which makes it into a Lie bialgebra; in the following we shall also retain from [loc. cit.] all the notation we need: in particular, we denote by Q, resp. P, the root lattice, resp. the weight lattice, of $\mathfrak{g}$, and by r the rank of $\mathfrak{g}$. In particular, when $\tau = 0$ we have the standard Sklyanin–Drinfeld cobracket. Similarly, $\mathfrak{g}$ may be any untwisted affine Kac–Moody algebra, as in [11] (with corresponding notation).

Now set $q := \exp(h)$; then $\mathbb{K}(q)$ is a subring of $\mathbb{K}[[\bar{h}]]$, hence also all its subrings are. Let $U_q(\mathfrak{g})$ be the Jimbo–Lusztig's quantum group over $\mathbb{K}(q)$, defined as $U_q(\mathfrak{g}) := U^Q_{q,\varphi}(\mathfrak{g})$ as in [10, Section 3.3], if $\mathfrak{g}$ is finite, and as $U_q(\mathfrak{g}) := U^Q_q(\mathfrak{g})$ as in [11, Section 3.3], if $\mathfrak{g}$ is affine. Furthermore, let $\hat{U}_q(\mathfrak{g})$ be the integer form of $U_q(\mathfrak{g})$ defined as $\hat{U}_q(\mathfrak{g}) := \mathfrak{U}^Q_\varphi(\mathfrak{g})$ (over $A := \mathbb{K}[q, q^{-1}]$) as in [10, Section 3.3], if $\mathfrak{g}$ is finite, and as $\hat{U}_q(\mathfrak{g}) := \mathfrak{U}^Q(\mathfrak{g})$ (over the ring A of rational functions in q having no poles at roots of unity of odd order) as in [11, Section 3.3], if $\mathfrak{g}$ is affine. In both cases A is a subring of $\mathbb{K}(q)$, hence of $\mathbb{K}[[\bar{h}]]$, thus we can define

$$U_{\bar{h}}(\mathfrak{g}) := \text{(separated)}\bar{h}\text{-adic completion of } \mathbb{K}[[\bar{h}]] \otimes_A \hat{U}_q(\mathfrak{g}). \tag{3.3}$$

It is well known that $\hat{U}_q(\mathfrak{g})/(q-1)\hat{U}_q(\mathfrak{g}) \cong U(\mathfrak{g})$: this and (3.3) imply that $U_{\bar{h}}(\mathfrak{g})$ has semiclassical limit $U(\mathfrak{g})$, thus it is a QUEA. In fact, $U_{\bar{h}}(\mathfrak{g})$ is the well known Drinfeld quantum group over $\mathbb{K}[[\bar{h}]]$, as defined in [2, Section 6]. In addition, let also $\tilde{U}_q(\mathfrak{g})$ be the integer form of $U_q(\mathfrak{g})$ defined as $\tilde{U}_q(\mathfrak{g}) := \mathcal{U}^Q_\varphi(\mathfrak{g})$ (over $A := \mathbb{K}[q, q^{-1}]$) as in [10, Section 3.3], if $\mathfrak{g}$ is finite, and as $\tilde{U}_q(\mathfrak{g}) := \mathcal{U}^Q(\mathfrak{g})$ (over the ring A above) as in [11, Section 3.3], if $\mathfrak{g}$ is affine.

Similarly, we do the same for the dual Lie bialgebra $\mathfrak{g}^*$ (denoted $\mathfrak{h}$ in [loc. cit.]), following [10, Section 6]—in the finite case—or [12, Section 5]—in the affine case, thus getting $U_q(\mathfrak{g}^*)$, $\hat{U}_q(\mathfrak{g}^*)$, $\tilde{U}_q(\mathfrak{g}^*)$, and $U_{\bar{h}}(\mathfrak{g}^*)$, the last one being a QUEA with $U(\mathfrak{g}^*)$ as semiclassical limit. From the description in [10,11], one sees that these objects are quite similar to the corresponding ones related to $\mathfrak{g}$.

Now consider $\hat{U}_q(\mathfrak{g})^* := \mathrm{Hom}_A(\hat{U}_q(\mathfrak{g}), A)$; from [10,11] we have the identification $\hat{U}^*_q(\mathfrak{g}) \cong \tilde{U}_q(\mathfrak{g}^*)$, and also $\tilde{U}_q(\mathfrak{g}^*) \overset{q\to 1}{\to} \tilde{U}_q(\mathfrak{g}^*)/(q-1)\tilde{U}_q(\mathfrak{g}^*) \cong F[[\mathfrak{g}]]$. Thus letting

$$F_{\bar{h}}[[\mathfrak{g}]] := \text{(separated) } \bar{h}\text{-adic completion of } \mathbb{K}[[\bar{h}]] \otimes_A \tilde{U}_q(\mathfrak{g}^*) \tag{3.4}$$

we have that $F_{\bar{h}}[[\mathfrak{g}]]$ is a QFSHA, with semiclassical limit $F[[\mathfrak{g}]]$.

The natural Hopf pairing $\langle , \rangle : \tilde{U}_q(\mathfrak{g}^*) \times \hat{U}_q(\mathfrak{g}) \to A$ yields a Hopf pairing $\langle , \rangle : F_{\bar{h}}[[\mathfrak{g}]] \times U_{\bar{h}}(\mathfrak{g}) \to \mathbb{K}[[\bar{h}]]$; moreover, it extends similarly to a perfect pairing $\langle , \rangle : U_q(\mathfrak{g}^*) \times U_q(\mathfrak{g}) \to \mathbb{K}(q)$. The analysis in [10,11] shows that

$$\tilde{U}_q(\mathfrak{g}) = (\hat{U}_q(\mathfrak{g}^*))^\circ := \{y \in U_q(\mathfrak{g}) | \langle \hat{U}_q(\mathfrak{g}^*), y \rangle \subseteq A\}. \tag{3.5}$$

In addition, by Proposition 1.4 we have also

$$U_{\overline{h}}(\mathfrak{g})' = (F_{\overline{h}}[[\mathfrak{g}]]^\vee)^\circ := \{y \in {}_F U_{\overline{h}}(\mathfrak{g}) \,|\, \langle F_{\overline{h}}[[\mathfrak{g}]]^\vee, y\rangle \subseteq \mathbb{K}[[\overline{h}]]\},$$

where we consider $\langle , \rangle : {}^F F_{\overline{h}}[[\mathfrak{g}]] \times {}^F U_{\overline{h}}(\mathfrak{g}) \to \mathbb{K}((h))$ to be the obvious pairing obtained by scalar extension from $\langle , \rangle : F_{\overline{h}}[[\mathfrak{g}]] \times U_{\overline{h}}(\mathfrak{g}) \to \mathbb{K}[[\overline{h}]]$.

Now, the very definitions of all the objects involved yield (via the analysis in [10,11])

$$\begin{aligned} F_{\overline{h}}[[\mathfrak{g}]]^\vee &= ((\text{separated})\overline{h}\text{-adic completion of } \mathbb{K}[[\overline{h}]] \otimes_A \tilde{U}_q(\mathfrak{g}^*))^\vee \\ &= (\text{separated})\overline{h}\text{-adic completion of } \mathbb{K}[[\overline{h}]] \otimes_A \hat{U}_q(\mathfrak{g}^*) = U_{\overline{h}}(\mathfrak{g}^*). \end{aligned}$$

This and (3.5) together give

$$U_{\overline{h}}(\mathfrak{g})' = (\text{separated})\overline{h}\text{-adic completion of } \mathbb{K}[[\overline{h}]] \otimes_A \tilde{U}_q(\mathfrak{g}). \tag{3.6}$$

This gives us a concrete description of $U_{\overline{h}}(\mathfrak{g})'$. If $U_{\overline{h}}(\mathfrak{g})$ is topologically generated—as usual—by Chevalley-like generators F_i, H_j, E_i (for i and j in some set of indices I and J, depending on the type of $\mathfrak{g}$) and if the F_α's, resp. E_α's, are (quantum) root vectors attached to the positive, resp. negative, roots of $\mathfrak{g}$ (like, for instance, in [10,11]), then $U_{\overline{h}}(\mathfrak{g})'$ is the unital topological subalgebra of $U_{\overline{h}}(\mathfrak{g})$ topologically generated by the set $\{\dot{F}_\alpha, \dot{E}_\alpha\}_\alpha \cup \{\dot{H}_j\}_j$ with $\dot{F}_\alpha := \overline{h} F_\alpha$, $\dot{E}_\alpha := \overline{h} E_\alpha$, $\dot{H}_j := \overline{h} H_j$ for all α and all j.

Having this description in our hands, we can recognize that Theorem 2.3 in this case is also proved in [8, Theorem 4.4] (or simply Corollary 3.8, for $c = 1$), for the finite case, and in [9], Corollary 2.5 (b), for the affine case.

4. Braidings from geometric quantization: Weinstein and Xu's approach

4.1. *The (global) classical $\mathcal{R}$-matrix (cf. [17])*

In this section we recall from [17] the construction of the global $\mathcal{R}$-matrix and point out how it provides a braiding.

From now on, let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Let $(\mathfrak{g}, \boldsymbol{r})$ be a (finite-dimensional) quasitriangular Lie bialgebra, and write $\boldsymbol{r} = \sum_i r_i^+ \otimes r_i^- \in \mathfrak{g} \otimes \mathfrak{g}$. Define linear maps

$$\boldsymbol{r}_\pm : \mathfrak{g}^* \to \mathfrak{g}^{**} = \mathfrak{g}, \qquad \boldsymbol{r}_\pm(\eta) := \pm \sum_i \eta(r_i^\pm) \cdot r_i^\mp \quad \forall \eta \in \mathfrak{g}^*. \tag{4.1}$$

These are both Lie algebra homomorphisms; if $(\mathfrak{g}, \boldsymbol{r})$ is triangular, then $\boldsymbol{r}_+ = \boldsymbol{r}_-$.

Let G be a complete Poisson–Lie group, and assume a dual Poisson–Lie group G^* exists; in general, *only a germ* of such a group is defined; then their tangent Lie bialgebras $\mathfrak{g}$ and $\mathfrak{g}^*$ are dual to each other. We say that *G is quasitriangular* if $\mathfrak{g}$ is quasitriangular *and* if the Lie algebra homomorphisms $\boldsymbol{r}_\pm : \mathfrak{g}^* \to \mathfrak{g}$ defined above *lift* to Lie group homomorphisms $R_\pm : G^* \to G$. In this case, we define

$$\phi\psi : G^* \to G, \qquad \phi(x) := R_+(x^{-1}), \qquad \psi(x) := R_-(x^{-1}) \quad \forall x \in G^*. \tag{4.2}$$

These are both Poisson morphisms; if G is triangular (i.e. the like is true for $\mathfrak{g}$) then $R_+ = R_-$, hence $\phi = \psi$.

We shall use the following conventions for *dressing transformations*. The left and right dressing transformation of G on G^* are denoted, respectively, by $\lambda_g u$ and $\rho_g u$ for all $g \in G$ and $u \in G^*$. Similarly, we denote the left and right dressing transformation of G^* on G by $\lambda_u g$ and $\rho_u g$ for all $u \in G^*$ and $g \in G$.

By definition, the *global classical $\mathcal{R}$–matrix* is

$$\mathcal{R} := \{(\psi(v^{-1}), u, \phi(\lambda_{\psi(v^{-1})} u), v) | u, v \in G^*\} = \{(\psi(v^{-1}), u, \rho_{v^{-1}} \phi(u), v) | u, v \in G^*\}$$

which is a Lagrangian submanifold of $D \times D$. It is shown in [17], how this object enjoys a bunch of properties which are exactly the analogous of those of a quantum R-matrix; in addition, if G is *triangular*, then $\mathcal{R}$ is *unitary*, by which we mean that $\mathcal{R}^{\mathrm{op}} = \mathcal{R}^{-1}$ (in the sense of [17], Remark 8.3). Moreover, these properties imply the following result.

Theorem 4.1 (cf. [19, Corollary 7.2]). *If G is a complete quasitriangular Poisson–Lie group, then the map $\mathcal{R} = \mathcal{R}_{\mathrm{WX}} : G^* \times G^* \to G^* \times G^*$ given by*

$$(u, v) \mapsto (\lambda_{\psi(v^{-1})} u, \lambda_{\phi(\lambda_{\psi(v^{-1})} u)} v) = (\lambda_{\psi(v^{-1})} u, \rho_{\phi(u^{-1})} v) \quad \forall\, u, v \in G^*$$

is a Poisson diffeomorphism such that

$$m \circ \mathcal{R} = m^{\mathrm{op}}, \qquad \mathcal{R} \circ (m \otimes \mathrm{id}) = (m \otimes \mathrm{id}) \circ \mathcal{R}_{23} \circ \mathcal{R}_{13},$$
$$\mathcal{R} \circ (\mathrm{id} \otimes m) = (\mathrm{id} \otimes m) \circ \mathcal{R}_{12} \circ \mathcal{R}_{13}, \tag{4.3}$$

where m is the product of G^* and $m^{\mathrm{op}} := m \circ \sigma$ (with σ as in Section 1.8). *In particular, $\mathcal{R}$ is a solution of the QYBE, and it restricts to a similar mapping $\mathcal{S} \times \mathcal{S} \to \mathcal{S} \times \mathcal{S}$ for every symplectic leaf $\mathcal{S}$ of G^*. In addition, if G is triangular then $\mathcal{R}$ is unitary, which means $\mathcal{R}^{-1} = \sigma \circ \mathcal{R} \circ \sigma$.*

Proof. It is just a matter of recalling or reformulating some results of [17]. The identity in the first line of (4.3) is proved by Theorem 5.1 in [loc. cit.]; the second line of identities instead is a simple reformulation of Theorem 5.4 in [loc.cit.]; finally, in the triangular case the unitarity of $\mathcal{R}$ follows from the unitarity of the global $\mathcal{R}$-matrix $\mathcal{R}$, by Corollary 8.2 and Remark 8.3 in [loc. cit.]. □

Corollary 4.2. *The mapping*

$$\mathfrak{R}_{\mathrm{WX}} := \mathcal{R}^* : F[G^*] \otimes F[G^*] = F[G^* \times G^*] \to F[G^* \times G^*] = F[G^*] \otimes F[G^*]$$

naturally induced by $\mathcal{R}$ is a braiding, which is unitary if $\mathcal{R}$ is. In particular, this canonically induces a braiding $\mathfrak{R}_{\mathrm{WX}} : F[[\mathfrak{g}^* \oplus \mathfrak{g}^*]] \to F[[\mathfrak{g}^* \oplus \mathfrak{g}^*]]$. Furthermore, the associated infinitesimal braiding $\bar{\mathfrak{R}}_{\mathrm{WX}} : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$ (cf. Section 3.1*) is trivial, i.e. $\bar{\mathfrak{R}}_{\mathrm{WX}} = \mathrm{id}_{\mathfrak{g} \oplus \mathfrak{g}}$.*

Proof. The first part of the claim—$\mathfrak{R}_{\mathrm{WX}}$ being a braiding, unitary if $\mathcal{R}_{\mathrm{WX}}$ is—follows trivially from Theorem 4.1 by duality; then $\mathfrak{R}_{\mathrm{WX}}$ automatically induces an infinitesimal braiding $\bar{\mathfrak{R}}_{\mathrm{WX}}$ as well.

To prove the second part—that is, $\bar{\mathfrak{R}}_{\mathrm{WX}}$ being trivial—we must go back to the definition and the properties of dressing actions. Recall that the left dressing action of G on G^* is defined as follows. For all $g \in G$, $\gamma \in G^*$, there exist unique $g^\gamma \in G$, $\gamma^g \in G^*$ such that $g \cdot \gamma = \gamma^g \cdot g^\gamma$; then the left dressing action $\lambda : G \times G^* \to G^*$ of G on G^* is given by $\lambda_g(\gamma) \equiv \lambda(g, \gamma) := \gamma^g$, for all $g \in G$, $\gamma \in G^*$.

Now, for all $X \in \mathfrak{g}$, $Y \in \mathfrak{g}^*$ and $t \in \mathbb{R}$, we have

$$\exp(tX)\exp(tY) = \exp(tY)^{\exp(tX)} \exp(t\,X)^{\exp(tY)},$$

whence Taylor expansion gives

$$\begin{aligned}\exp(tY)^{\exp(tX)} &= \left(1 + tY + t^2 \tfrac{1}{2}(Y^2) + \cdots\right)^{(1+tX+t^2X^2/2+\cdots)} \\ &= 1 + tY + t^2 Y^X + \cdots + t^2 \tfrac{1}{2}(Y^2) + \cdots,\end{aligned}$$

(where Y^X denotes the action of X onto Y induced at the infinitesimal level by the dressing action), hence at first order in t we have simply Y! Applied to the situation $\exp(tX) = \psi(v^{-1})$, $\exp(tY) = u$ this says that the first entry of $T_{(e,e)}(\mathcal{R}_{\mathrm{WX}})(Y, V)$ is just Y (here $V := \log(v)$, and e denotes the unit element of G^*). Similarly, carrying out a like analysis on the right dressing action we get that the second entry of $T_{(e,e)}(\mathcal{R}_{\mathrm{WX}})(Y, V)$ is simply V. Therefore, $T_{(e,e)}(\mathcal{R}_{\mathrm{WX}}) = \mathrm{id}_{\mathfrak{g}^*\oplus\mathfrak{g}^*}$. As $\bar{\mathfrak{R}}_{\mathrm{WX}}$ is just the dual of $T_{(e,e)}(\mathcal{R}_{\mathrm{WX}})$, it is trivial as well, q.e.d. □

4.2. *The factorizable case*

Let $(\mathfrak{g}, \boldsymbol{r})$ be a quasitriangular Lie bialgebra. If the bilinear form on $\mathfrak{g} \otimes \mathfrak{g}$ naturally associated to $\boldsymbol{r} + \boldsymbol{r}^{\mathrm{op}}$ is non-degenerate, then $(\mathfrak{g}, \boldsymbol{r})$ is said to be *factorizable*. In this case, the corresponding linear map $j := \boldsymbol{r}_+ - \boldsymbol{r}_- : \mathfrak{g}^* \to \mathfrak{g}$ is invertible. Now let G be a Poisson–Lie group corresponding to the Lie bialgebra above, and let G^* be its connected, simply connected Poisson dual. The Lie algebra morphisms $\boldsymbol{r}_\pm : \mathfrak{g}^* \to \mathfrak{g}$ lift to group morphisms $R_\pm : G^* \to G$, thus we may define the map $J : G^* \to G$ by $J(u) := R_+(u)R_-(u)^{-1}$ (for all $u \in G^*$) whose derivative at the identity element $u \in G^*$ is j (note that neither j nor J is a morphism). When J is a global diffeomorphism, we say that *the group G is factorizable*, since for each $g \in G$ we have the factorization $g = g_+ g_-^{-1}$, where $g_\pm := R_\pm(J^{-1}(g))$. Thanks to [19, Proposition 9.1], any connected, simply connected, factorizable Poisson–Lie group is complete.

Now, factorizability enables us to describe the classical $\mathcal{R}$-matrix quite explicitly.

Theorem 4.3 (cf. [19, Theorem 9.2]). *Let G be a factorizable Poisson–Lie group, and use $J : G^* \to G$ to identify G^* with G (hence also $G \times G$ with $G^* \times G^*$). Then*:

(a) *the (global) classical $\mathcal{R}$–matrix $\mathcal{R} \in (G \times G) \times (G \times G)$ takes the form*

$$\mathcal{R} = \{(y_-, x, (y_- x y_-^{-1})_+^{-1}, y) | \forall\, x, y \in G\};$$

(b) *the map $\mathcal{R} = \mathcal{R}_{\mathrm{WX}} : G \times G \to G \times G$ of Theorem 4.1 is given by*

$$\mathcal{R}_{\mathrm{WX}}(x, y) = (y_- x y_-^{-1}, (y_- x y_-^{-1})_+^{-1} y (y_- x y_-^{-1})_+^{+1}) \quad \forall\, (x, y) \in G \times G.$$

Remark 4.4. As we pointed out in the Introduction, one can carry over the construction of Weinstein and Xu in purely local terms, just performing it on the germ of Poisson group underlying the quasitriangular Lie bialgebra $(\mathfrak{g}, \boldsymbol{r})$, and eventually get a braiding $\mathfrak{R}_{\mathrm{WX}} : F[[\mathfrak{g}^* \oplus \mathfrak{g}^*]] \to F[[\mathfrak{g}^* \oplus \mathfrak{g}^*]]$ and an associated infinitesimal braiding $\bar{\mathfrak{R}}_{\mathrm{WX}} : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$. Our next result is that the latter is always trivial whenever $(\mathfrak{g}, \boldsymbol{r})$ is factorizable.

Proposition 4.5. *Let the quasitriangular Lie bialgebra* $(\mathfrak{g}, \boldsymbol{r})$ *be factorizable. Then the infinitesimal braiding* $\bar{\mathfrak{R}}_{\mathrm{WX}} : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$ *is trivial, i.e.* $\bar{\mathfrak{R}}_{\mathrm{WX}} = \mathrm{id}_{\mathfrak{g}} \oplus \mathfrak{g}$.

Proof. Let G_{loc} be the germ of Poisson group associated to the Lie bialgebra $\mathfrak{g}$. Then the "local" version of Theorem 4.3 (b) ensures that the map $\mathcal{R}_{\mathrm{WX}} : G_{\mathrm{loc}} \times G_{\mathrm{loc}} \to G_{\mathrm{loc}} \times G_{\mathrm{loc}}$ is given by $\mathcal{R}_{\mathrm{WX}}(x, y) = (y_- x y_-^{-1}, (y_- x y_-^{-1})_+^{-1} y (y_- x y_-^{-1})_+^{+1})$ for all $x, y \in G_{\mathrm{loc}}$. Now, for all $A, B \in \mathfrak{g}$ and $t \in \mathbb{R}$ we have

$$\begin{aligned} &\exp(tA)\exp(tB)\exp(tA)^{-1} \\ &\quad = \left(1 + tA + t^2\tfrac{1}{2}(A^2) + \cdots\right)\left(1 + tB + t^2\tfrac{1}{2}(B^2) + \cdots\right)\left(1 - tA + t^2\tfrac{1}{2}(A^2) - \cdots\right) \\ &\quad = 1 + tB + t^2\tfrac{1}{2}((2(AB - BA) + B^2)) + \cdots . \end{aligned}$$

Applying this recipe to $A = \log(y_-)$, $B = \log(x)$, and looking at first order (in t) we find out that the first entry of $T_{(e,e)}(\mathcal{R}_{\mathrm{WX}})(x, y)$ is just x; similarly we get that the second entry of $T_{(e,e)}(\mathcal{R}_{\mathrm{WX}})(x, y)$ is y. Thus $T_{(e,e)}(\mathcal{R}_{\mathrm{WX}})$ is the identity, and since $\bar{\mathfrak{R}}_{\mathrm{WX}}$ is just its dual, it is the identity as well, q.e.d. □

5. Comparing the braidings $\mathfrak{R}_{\mathrm{WX}}$ and $\mathfrak{R}_{\mathrm{GH}}$: the case of $\mathfrak{g} = \mathfrak{sl}_2$

5.1. *The general problem*

We noticed that the construction of [17] can be performed for any quasitriangular Lie bialgebra by acting locally, so to get a braiding $\mathfrak{R}_{\mathrm{WX}}$ on the dual formal Poisson group, exactly like one can do following [13] to get a braiding $\mathfrak{R}_{\mathrm{GH}}$. Since these braidings share similar properties—like functoriality and infinitesimal triviality, for instance—we are led to raise the following.

Question. *Given any quasitriangular Lie bialgebra* $\mathfrak{g}$, *do the braidings* $\mathfrak{R}_{\mathrm{WX}}$ *and* $\mathfrak{R}_{\mathrm{GH}}$ *on* $F[[\mathfrak{g}^*]]$ *coincide*?

The purpose of the present section is to provide a positive answer to this question for the simplest case of $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$. The general case is tackled and solved in [13].

5.2. *The geometrical setting*

In this section, let $\mathbb{K} = \mathbb{C}$. Let $G := \mathrm{SL}_2 \equiv \mathrm{SL}_2(\mathbb{C})$. Its tangent Lie algebra $\mathfrak{g} = \mathfrak{sl}_2$ is generated by f, h, e (the *Chevalley generators*) with relations $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$. The formulæ $\delta(f) = (f \otimes h - h \otimes f)/2$, $\delta(h) = 0$, $\delta(e) = (e \otimes h - h \otimes e)/2$, define a Lie cobracket on $\mathfrak{g}$. Indeed, this makes $\mathfrak{sl}_2$ into a quasitriangular Lie bialgebra, whose $\boldsymbol{r}$-matrix is $\boldsymbol{r} := e \otimes f + (h \otimes h)/4$. This corresponds to a structure of complex Poisson–Lie (actually, *algebraic*) group on G, which is complete and quasitriangular.

In the dual Lie bialgebra $\mathfrak{g}^* = \mathfrak{sl}_2^*$, let $\{e^*, f^*, h^*\}$ be the basis dual to $\{e, f, h\}$, and consider the basis $\{\mathrm{e} := e^*, \mathrm{f} := f^*, \mathrm{h} := -2h^*\}$. Then the Lie bialgebra structure of $\mathfrak{sl}_2^*$ is described by the formulæ $[\mathrm{h}, \mathrm{e}] = \mathrm{e}$, $[\mathrm{h}, \mathrm{f}] = \mathrm{f}$, $[\mathrm{e}, \mathrm{f}] = 0$, and $\delta(\mathrm{f}) = \mathrm{h} \otimes \mathrm{f} - \mathrm{f} \otimes \mathrm{h}$, $\delta(\mathrm{h}) = 2(\mathrm{f} \otimes \mathrm{e} - \mathrm{e} \otimes \mathrm{f})$, $\delta(\mathrm{e}) = \mathrm{e} \otimes \mathrm{h} - \mathrm{h} \otimes \mathrm{e}$. Then $\mathfrak{sl}_2^*$ can be realized as the Lie algebra of pairs of matrices

$$\mathfrak{sl}_2^* = \left\{ \left(\begin{pmatrix} -t & 0 \\ c & t \end{pmatrix}, \begin{pmatrix} t & b \\ 0 & -t \end{pmatrix} \right) \middle| \, b, c, t \in \mathbb{K} \right\} \subseteq \mathfrak{sl}_2 \times \mathfrak{sl}_2 \tag{5.1}$$

(with the Lie subalgebra structure inside $\mathfrak{sl}_2 \times \mathfrak{sl}_2$). It follows that the unique connected simply connected complex Poisson–Lie (actually, *algebraic*) group whose tangent Lie bialgebra is $\mathfrak{sl}_2^*$ can be realized as the group of pairs of matrices (the left subscript s meaning "simply connected")

$${}_s\mathrm{SL}_2^* = \left\{ \left(\begin{pmatrix} z^{-1} & 0 \\ y & z \end{pmatrix}, \begin{pmatrix} z & x \\ 0 & z^{-1} \end{pmatrix} \right) \middle| \, x, y \in \mathbb{K}, z \in \mathbb{K} \setminus \{0\} \right\} \subseteq \mathrm{SL}_2 \times \mathrm{SL}_2 \tag{5.2}$$

(with the subgroup structure inside $\mathrm{SL}_2 \times \mathrm{SL}_2$); this group has a "small" center, namely $Z := \{(I, I), (-I, -I)\}$, so there is only one other (Poisson) group sharing the same Lie (bi)algebra, namely the quotient ${}_a\mathrm{SL}_2^* := {}_s\mathrm{SL}_2^*/Z$ (the adjoint of ${}_s\mathrm{SL}_2^*$, as the left subscript a means). Therefore $F[{}_s\mathrm{SL}_2^*]$ is the unital associative commutative $\mathbb{K}$-algebra with generators $x, z^{\pm 1}, y$, with Poisson Hopf structure given by

$$\begin{aligned}
&\Delta(x) = x \otimes z^{-1} + z \otimes x, \qquad \Delta(z^{\pm 1}) = z^{\pm 1} \otimes z^{\pm 1}, \qquad \Delta(y) = y \otimes z^{-1} + z \otimes y, \\
&\epsilon(x) = 0, \qquad \epsilon(z^{\pm 1}) = 1, \qquad \epsilon(y) = 0, \\
&S(x) = -x, \qquad S(z^{\pm 1}) = z^{\mp 1}, \qquad S(y) = -y, \\
&\{x, y\} = z^{-2} - z^{+2}, \qquad \{z^{\pm 1}, x\} = \mp \tfrac{1}{2} x z^{\pm 1}, \qquad \{z^{\pm 1}, y\} = \pm \tfrac{1}{2} z^{\pm 1} y.
\end{aligned}$$

(N.B.: with respect to this presentation, we have $\mathrm{f} = \partial_y|_u$, $\mathrm{h} = (z/2)\partial_z|_u$, $\mathrm{e} = \partial_x|_u$, where u is the identity element of ${}_s\mathrm{SL}_2^*$). Moreover, $F[{}_a\mathrm{SL}_2^*]$ can be identified with the Poisson Hopf subalgebra of $F[{}_s\mathrm{SL}_2^*]$ spanned by products of an even number of generators, i.e. monomials of even degree. This is generated as a unital subalgebra, by xz, $z^{\pm 2}$, and $z^{-1}y$. Finally, the (algebra of regular functions on the) Poisson algebraic formal group $F[[\mathfrak{sl}_2^*]]$ is the $\mathrm{Ker}(\epsilon)$–adic completion of both $F[{}_s\mathrm{SL}_2^*]$ and $F[{}_a\mathrm{SL}_2^*]$; in the first case $\mathrm{Ker}(\epsilon)$ is generated (as an ideal) by x, $(z^{\pm 1} - 1)$ and y, therefore $F[[\mathfrak{sl}_2^*]] = \mathbb{K}[[x, (z-1), y]]$ as a topological $\mathbb{K}$-algebra (note that $z^{-1} - 1 = \sum_{n>0} (-1)^n (z-1)^n$, so the generator $z^{-1} - 1$ is superfluous) with the unique Poisson Hopf structure which extends by continuity the one on $F[{}_s\mathrm{SL}_2^*]$.

5.3. *Weinstein and Xu's construction*

In the framework of Section 5.2 let $G := \mathrm{SL}_2$, $G^* := {}_s\mathrm{SL}_2^*$. In this section we compute the braiding $\mathfrak{R}_{\mathrm{WX}}$ for G; despite the fact that not all requirements of [17] are fulfiled, we can show that that construction can still be carried out at the local level: to fulfill this goal is then just a matter of matrix computation.

It follows from definitions—cf. [19, Section 9]—that the maps $\boldsymbol{r}_\pm : \mathfrak{g}^* \to \mathfrak{g}$ are given by $\boldsymbol{r}_+(\mathrm{e}) = 2\mathrm{f}$, $\boldsymbol{r}_+(\mathrm{h}) = -h/2$, $\boldsymbol{r}_+(\mathrm{f}) = 0$, $\boldsymbol{r}_-(\mathrm{e}) = 0$, $\boldsymbol{r}_-(\mathrm{h}) = +h/2$, $\boldsymbol{r}_-(\mathrm{f}) = -2e$, and the maps $R_\pm : G^* \to G$ are, respectively, the projection to the second and the first factor w.r.t. to the description of $G^* = {}_s\mathrm{SL}_2^*$ in (5.2). Then for the maps $j : \mathfrak{g}^* \to \mathfrak{g}$ and $J : G^* \to G$ defined in Section 4.2 we have that j is bijective but J is not, for it has kernel $\mathrm{Ker}(J) = Z$ (hence it is a 2–to–1 map) and image

$$\mathrm{Im}(J) = G^0 := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in \mathbb{C},\ d \neq 0 \right\}$$

that is the *big cell* of $G = \mathrm{SL}_2$: in fact, J is an unramified 2-fold covering of G^0. Therefore, J is not a global diffeomorphism, but it factors to a global diffeomorphism

$$J_a : {}_aG^* \equiv {}_a\mathrm{SL}_2^* := \frac{{}_s\mathrm{SL}_2^*}{Z} \stackrel{\simeq}{\to} G^0$$

given by $J_a(g \cdot Z) := J(g)$ for all $g \in {}_s\mathrm{SL}_2^*$. We need a section of J and of J_a. Since

$$J\left(\begin{pmatrix} A^{-1} & 0 \\ B & A^{+1} \end{pmatrix}, \begin{pmatrix} A^{+1} & C \\ 0 & A^{-1} \end{pmatrix} \right) = \begin{pmatrix} A^{+1} & C \\ 0 & A^{-1} \end{pmatrix} \cdot \begin{pmatrix} A^{+1} & 0 \\ -B & A^{-1} \end{pmatrix} = \begin{pmatrix} A^{+2} - BC & A^{-1}C \\ -A^{-1}B & A^{-2} \end{pmatrix}$$

we have $J\left(\begin{pmatrix} A^{-1} & 0 \\ B & A^{+1} \end{pmatrix}, \begin{pmatrix} A^{+1} & C \\ 0 & A^{-1} \end{pmatrix} \right) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ if and only if

$$A = \pm d^{-1/2}, \qquad B = \pm b d^{-1/2}, \qquad C = \mp c d^{-1/2} \tag{5.3}$$

for any matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G^0$; these formulæ clearly define two differentiable sections of J (taking either upper or lower signs) and one of J_a (for which the sign is irrelevant).

Remark. Although G is not factorizable, nevertheless we can still use Theorem 4.3 (b) to compute the map $\mathcal{R}_{\mathrm{WX}}$, namely

$$\mathcal{R}_{\mathrm{WX}}(X', Y') = (J^{-1}(Y_- X Y_-^{-1}), J^{-1}((Y_- X Y_-^{-1})_+^{-1} Y (Y_- X Y_-^{-1})_+^{+1})) \tag{5.4}$$

for all $(X', Y') \in G^* \times G^*$ and $(X, Y) := (J(X'), J(Y')) \in G \times G$, where J^{-1} is one of the two aforesaid sections of J, namely the unique one such that the resulting $\mathcal{R}_{\mathrm{WX}}(X', Y')$ map (e_{G^*}, e_{G^*}) onto itself. In fact, although J is not a diffeomorphism it is nevertheless a (finite) covering on G^0, hence it is a *local diffeomorphism* (around the identity element $e_{G^*} \in G^*$) on G^0, therefore the description of $\mathcal{R}_{\mathrm{WX}}(X', Y')$ afforded by Theorem 4.3 (b), through J and a local section J^{-1}, is still available (locally around $(e_{G^*}, e_{G^*}) \in G^* \times G^*$).

To have a *global* description, one has just to choose the unique section J^{-1} which maps $e_{G^0} = e_G$ onto e_{G^*}. Therefore, we shall now go on computing $\mathcal{R}_{\mathrm{WX}}$ following this strategy.

Let $X := J\left(\begin{pmatrix} z^{-1} & 0 \\ y & z \end{pmatrix}, \begin{pmatrix} z & x \\ 0 & z^{-1} \end{pmatrix}\right)$, $Y := J\left(\begin{pmatrix} \zeta^{-1} & 0 \\ \eta & \zeta \end{pmatrix}, \begin{pmatrix} \zeta & \chi \\ 0 & \zeta^{-1} \end{pmatrix}\right) \in J(G^*)$ $= G^0$. Then we have

$$
\begin{aligned}
Y_- \cdot X \cdot Y_-^{-1} &= \begin{pmatrix} \zeta^{-1} & 0 \\ \eta & \zeta \end{pmatrix}\begin{pmatrix} z & x \\ 0 & z^{-1} \end{pmatrix}\begin{pmatrix} z & 0 \\ -y & z^{-1} \end{pmatrix}\begin{pmatrix} \zeta & 0 \\ -\eta & \zeta^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \zeta^{-1} & 0 \\ \eta & \zeta \end{pmatrix}\begin{pmatrix} z^2 - xy & z^{-1}x \\ -z^{-1}y & z^{-2} \end{pmatrix}\begin{pmatrix} \zeta & 0 \\ -\eta & \zeta^{-1} \end{pmatrix} \\
&= \begin{pmatrix} z^2 - xy - \eta\zeta^{-1}xz^{-1} & \zeta^{-2}z^{-1}x \\ \eta\zeta z^2 - (\eta\zeta^{-2}\zeta + yz^{-1}\zeta^{+2})\Theta^2 & z^{-2}\Theta^2 \end{pmatrix}
\end{aligned}
$$

with $\Theta := (1 + \eta xz\zeta^{-1})^{1/2}$. Using (5.3) we get from this

$$
\begin{aligned}
(Y_-XY_-^{-1})_+ &= \pm \begin{pmatrix} z^{+1}\Theta^{-1} & x\zeta^{-2}\Theta^{-1} \\ 0 & z^{-1}\Theta^{+1} \end{pmatrix} \\
(Y_-XY_-^{-1})_- &= \pm \begin{pmatrix} z^{-1}\Theta^{+1} & 0 \\ y\zeta^{+2}\Theta^{+1} + \eta\zeta(z^{-1}\Theta^{+1} - z^{+3}\Theta^{-1}) & z^{+1}\Theta^{-1} \end{pmatrix},
\end{aligned}
\tag{5.5}
$$

which gives

$$
\begin{aligned}
J^{-1}(Y_-XY_-^{-1}) = \pm \Bigg(\begin{pmatrix} z^{-1}\Theta & 0 \\ y\zeta^2\Theta + \eta\zeta(z^{-1}\Theta - z^3\Theta^{-1}) & z\Theta^{-1} \end{pmatrix}, \\
\times \begin{pmatrix} z\Theta^{-1} & x\zeta^{-2}\Theta^{-1} \\ 0 & z^{-1}\Theta^{+1} \end{pmatrix} \Bigg)
\end{aligned}
\tag{5.6}
$$

as possible preimages of $Y_-XY_-^{-1}$ in $G^* \times G^*$. This takes care of the first entry in the right-hand-side of (5.4).

As for the second entry, we have (noting that the ambiguity of sign in (5.5) is irrelevant)

$$
\begin{aligned}
&(Y_-XY_-^{-1})_+^{-1} \cdot Y \cdot (Y_-XY_-^{-1})_+^{+1} \\
&= \begin{pmatrix} z^{-1}\Theta^{+1} & -x\zeta^{-2}\Theta^{-1} \\ 0 & z^{+1}\Theta^{-1} \end{pmatrix} \cdot \begin{pmatrix} \zeta^{+1} & \chi \\ 0 & \zeta^{-1} \end{pmatrix} \\
&\quad \cdot \begin{pmatrix} \zeta^{+1} & 0 \\ -\eta & \zeta^{-1} \end{pmatrix} \cdot \begin{pmatrix} z^{-1}\Theta^{+1} & -x\zeta^{-2}\Theta^{-1} \\ 0 & z^{+1}\Theta^{-1} \end{pmatrix} \\
&= \begin{pmatrix} z^{-1}\Theta^{+1} & -x\zeta^{-2}\Theta^{-1} \\ 0 & z^{+1}\Theta^{-1} \end{pmatrix} \cdot \begin{pmatrix} \zeta^{+2} - \eta\chi & \chi\zeta^{-1} \\ -\eta\zeta^{-1} & \zeta^{-2} \end{pmatrix} \cdot \begin{pmatrix} z^{-1}\Theta^{+1} & -x\zeta^{-2}\Theta^{-1} \\ 0 & z^{+1}\Theta^{-1} \end{pmatrix} \\
&= \begin{pmatrix} \zeta^{+2} - \eta\chi + \eta xz^{+1}\zeta^{-3}\Theta^{-2} & xz^{-1} + \chi z^{-2}\zeta^{-1} - xz^{-1}\zeta^{-4}\Theta^{-2} \\ -\eta z^{+2}\zeta^{-1}\Theta^{-2} & \zeta^{-2}\Theta^{-2} \end{pmatrix}.
\end{aligned}
$$

Again using (5.3) we find

$$
\begin{aligned}
&((Y_- X Y_-^{-1})_+^{-1} \cdot Y \cdot (Y_- X Y_-^{-1})_+^{+1})_+ \\
&\quad = \pm \begin{pmatrix} \zeta^{+1}\Theta^{+1} & \chi z^{-2}\Theta^{+1} + xz^{-1}\zeta^{+1}\Theta^{+1} - xz^{-1}\zeta^{-3}\Theta^{-1} \\ 0 & \zeta^{-1}\Theta^{-1} \end{pmatrix}, \\
&((Y_- X Y_-^{-1})_+^{-1} \cdot Y \cdot (Y_- X Y_-^{-1})_+^{+1})_- \\
&\quad = \pm \begin{pmatrix} \zeta^{-1}\Theta^{-1} & 0 \\ \eta z^{+2}\Theta^{-1} & \zeta^{+1}\Theta^{+1} \end{pmatrix},
\end{aligned} \tag{5.7}
$$

which gives

$$
\begin{aligned}
&J^{-1}((Y_- X Y_-^{-1})_+^{-1} \cdot Y \cdot (Y_- X Y_-^{-1})_+^{+1}) \\
&\quad = \pm \left(\begin{pmatrix} \zeta^{+1}\Theta^{+1} & \chi z^{-2}\Theta^{+1} + xz^{-1}\zeta^{+1}\Theta^{+1} - xz^{-1}\zeta^{-3}\Theta^{-1} \\ 0 & \zeta^{-1}\Theta^{-1} \end{pmatrix}, \right. \\
&\qquad \left. \times \begin{pmatrix} \zeta^{-1}\Theta^{-1} & 0 \\ \eta z^{+2}\Theta^{-1} & \zeta^{+1}\Theta^{+1} \end{pmatrix} \right)
\end{aligned}
$$

as possible preimages of $(Y_- X Y_-^{-1})_+^{-1} \cdot Y \cdot (Y_- X Y_-^{-1})_+^{+1}$ in $G^* \times G^*$. This takes care of the second entry in the right-hand-side of (5.4). Finally, imposing the condition $\mathfrak{R}_{\mathrm{WX}}(e_{G^*}, e_{G^*}) = (e_{G^*}, e_{G^*})$ we must always take the "plus" signs, here and in (5.6).

Using notation $x_1 := x \otimes 1$, $z_1^{\pm 1} := z^{\pm 1} \otimes 1$, $y_1 := y \otimes 1$, $x_2 := 1 \otimes x$, $z_2^{\pm 1} := 1 \otimes z^{\pm 1}$ and $y_2 := 1 \otimes y$ we see that these last formulæ together with (5.6) give

$$
\begin{aligned}
&\mathfrak{R}_{\mathrm{WX}}(x_1) = x_1 \cdot z_2^{-2} \cdot \Theta^{-1}, \qquad \mathfrak{R}_{\mathrm{WX}}(z_1^{\pm 1}) = z_1^{\pm 1} \cdot \Theta^{\mp 1}, \\
&\mathfrak{R}_{\mathrm{WX}}(y_1) = y_1 \cdot z_2^{+2} \cdot \Theta^{+1} + y_2 \cdot z_2^{+1} z_1^{-1} \cdot \Theta^{+1} - y_2 \cdot z_2^{+1} z_1^{+3} \cdot \Theta^{-1}, \\
&\mathfrak{R}_{\mathrm{WX}}(x_2) = x_2 \cdot z_1^{-2} \cdot \Theta^{+1} + x_1 \cdot z_1^{-1} z_2^{+1} \cdot \Theta^{+1} - x_1 \cdot z_1^{-1} z_2^{-3} \cdot \Theta^{-1}, \\
&\mathfrak{R}_{\mathrm{WX}}(z_2^{\pm 1}) = z_2^{\pm 1} \cdot \Theta^{\pm 1}, \qquad \mathfrak{R}_{\mathrm{WX}}(y_2) = y_2 \cdot z_1^{+2} \cdot \Theta^{-1}
\end{aligned} \tag{5.8}
$$

for the braiding $\mathfrak{R}_{\mathrm{WX}}$. To summarize, our discussion lead to the following result (which somewhat improves the analysis of the like problem performed in [19, Section 9.7]).

Theorem 5.1. *Let* $({}_s\mathrm{SL}_2^* \times {}_s\mathrm{SL}_2^*)_2$ *be a 2-fold covering of* ${}_s\mathrm{SL}_2^* \times {}_s\mathrm{SL}_2^*$ *and let* $({}_s\mathrm{SL}_2^* \times {}_s\mathrm{SL}_2^*)_2^{(\Theta)} := ({}_s\mathrm{SL}_2^* \times {}_s\mathrm{SL}_2^*)_2 \setminus \{\Theta \neq 0\}$ *(a Zarisky open subset of* ${}_s\mathrm{SL}_2^* \times {}_s\mathrm{SL}_2^*$*).*

Then the map $\mathcal{R}_{\mathrm{WX}}$ *is a Poisson diffeomorphism from* $({}_s\mathrm{SL}_2^* \times {}_s\mathrm{SL}_2^*)_2^{(\Theta)}$ *to itself.*

In addition, $\mathcal{R}_{\mathrm{WX}}$ *is well defined also on a distinguished variety* $({}_a\mathrm{SL}_2^* \times {}_a\mathrm{SL}_2^*)_2^{(\Theta)}$ *which is a 2-fold covering of* ${}_a\mathrm{SL}_2^* \times {}_a\mathrm{SL}_2^*$ *minus one distinguished divisor. In terms of function algebras, these diffeomorphisms are uniquely determined by formulæ* (5.8), which also define the braiding $\mathfrak{R}_{\mathrm{WX}} : F[[\mathfrak{sl}_2^* \oplus \mathfrak{sl}_2^*]] \xrightarrow{\cong} F[[\mathfrak{sl}_2^* \oplus \mathfrak{sl}_2^*]]$.

5.4. The quantization deformation construction

(Warning: in the present section we follow the lines of [8], but we adopt *different normalizations* in the definition of quantum groups and their R-matrices.) Let $U_{\overline{h}}(\mathfrak{g}) = U_{\overline{h}}(\mathfrak{sl}_2)$ be the unital associative topological $\mathbb{K}[[\overline{h}]]$-algebra with (topological) generators X, H, Y, and relations

$$HX - XH = +2X, \qquad HY - YH = -2Y, \qquad XY - YX = \frac{e^{+\overline{h}H/2} - e^{-\overline{h}H/2}}{e^{+\overline{h}/2} - e^{-\overline{h}/2}}. \tag{5.9}$$

For later use we set also $L^{\pm 1} := e^{\pm\overline{h}H/4}$ and $q^{\pm 1} := e^{\pm\overline{h}/2}$; therefore

$$L^{\pm 1}X = q^{\pm 1}XL^{\pm 1}, \qquad L^{\pm 1}Y = q^{\mp 1}YL^{\pm 1}, \qquad XY - YX = \frac{L^{+2} - L^{-2}}{q^{+1} - q^{-1}}. \tag{5.10}$$

There is a Hopf algebra structure on $U_{\overline{h}}(\mathfrak{sl}_2)$, given on generators by

$$\begin{aligned}
&\Delta(X) = X \otimes e^{+\overline{h}H/4} + e^{-\overline{h}H/4} \otimes X = X \otimes L^{+1} + L^{-1} \otimes X,\\
&\Delta(H) = H \otimes 1 + 1 \otimes H,\\
&\Delta(Y) = Y \otimes e^{+\overline{h}H/4} + e^{-\overline{h}H/4} \otimes Y = Y \otimes L^{+1} + L^{-1} \otimes Y,\\
&\epsilon(X) = \epsilon(H) = \epsilon(Y) = 0, \qquad \epsilon(L^{\pm 1}) = 1, \qquad S(X) = -e^{-\overline{h}/2}X = -q^{-1}X,\\
&S(H) = -H, \qquad S(Y) = -e^{-\overline{h}/2}Y = -q^{-1}Y, \qquad S(L^{\pm 1}) = L^{\mp 1}.
\end{aligned}$$

Then $U_{\overline{h}}(\mathfrak{sl}_2)$ is a QUEA, whose semiclassical limit is $U(\mathfrak{sl}_2)$ (w.r.t. the co-Poisson structure considered in Section 5.2). For later use we record that

$$\{X^a H^b Y^c \mid a, b, c \in \mathbb{N}\} \quad \text{is a topological } \mathbb{K}[[\overline{h}]]\text{-basis of } U_{\overline{h}}(\mathfrak{sl}_2). \tag{5.11}$$

The very definitions also show that the unital subalgebra of $U_{\overline{h}}(\mathfrak{sl}_2)$ generated over the Laurent polynomial ring $\mathbb{K}[q, q^{-1}]$ by X, $L^{\pm 1}D := (L-1)/(q-1)$, $\Gamma := (L^{+2} - L^{-2})/(q^{+1} - q^{-1})$ and Y is a Hopf algebra (over $\mathbb{K}[q, q^{-1}]$) as well, which we denote by $U_q^s(\mathfrak{sl}_2)$. Similarly, the unital subalgebra of $U_{\overline{h}}(\mathfrak{sl}_2)$ generated over the Laurent polynomial ring $\mathbb{K}[q, q^{-1}]$ by XL^{-1}, $K^{\pm 1} := L^{\pm 2}$, $T := (K-1)/(q-1)$, $\Gamma := (K^{+1} - K^{-1})/(q^{+1} - q^{-1})$ and $L^{+1}Y$ is a Hopf algebra as well (a Hopf $\mathbb{K}[q, q^{-1}]$–subalgebra of $U_q^s(\mathfrak{sl}_2)$), which we denote by $U_q^a(\mathfrak{sl}_2)$.

Now we go and compute $U_{\overline{h}}(\mathfrak{sl}_2)'$. From definitions we get, for any $n \in \mathbb{N}$,

$$\begin{aligned}
\delta_n(X) &= (\mathrm{id} - \epsilon)^{\otimes n}(\Delta^n(X)) = (\mathrm{id} - \epsilon)^{\otimes n}\left(\sum_{s=1}^{n} (L^{-1})^{\otimes(s-1)} \otimes X \otimes (L^{+1})^{\otimes(n-s)}\right)\\
&= \sum_{s=1}^{n}\left(\sum_{t>0}(-\overline{h})^t \frac{H^t}{t!}\right)^{\otimes(s-1)} \otimes X \otimes \left(\sum_{r>0}(+\overline{h})^r \frac{H^r}{r!}\right)^{\otimes(n-s)}\\
&\in \overline{h}^{n-1}U_{\overline{h}}(\mathfrak{sl}_2) \setminus \overline{h}^{n}U_{\overline{h}}(\mathfrak{sl}_2)
\end{aligned}$$

from which we get $\dot{X} := \overline{h}X \in U_{\overline{h}}(\mathfrak{g})' \setminus \overline{h}U_{\overline{h}}(\mathfrak{g})'$. Similarly $\dot{Y} := \overline{h}Y \in U_{\overline{h}}(\mathfrak{g})' \setminus \overline{h}U_{\overline{h}}(\mathfrak{g})'$. As for the generator H, we have $\Delta^n(H) = \sum_{s=1}^{n} 1^{\otimes(s-1)} \otimes H \otimes 1^{\otimes(n-s)}$ for all $n \in \mathbb{N}$, whence for $\delta_n = (\mathrm{id} - \epsilon)^{\otimes n} \circ \Delta^n$ we have

$$\delta_0(H) = 0, \quad \delta_1(H) = \; H \in U_{\overline{h}}(\mathfrak{g}) \setminus \overline{h} U_{\overline{h}}(\mathfrak{g}), \qquad \delta_n(H) = 0 \in \overline{h}^n U_{\overline{h}}(\mathfrak{sl}_2) \;\; \forall\, n > 1$$

so that $\dot{H} := \overline{h} H \in U_{\overline{h}}(\mathfrak{sl}_2)' \setminus \overline{h} U_{\overline{h}}(\mathfrak{sl}_2)'$. Therefore $U_{\overline{h}}(\mathfrak{sl}_2)'$ contains the subalgebra U' topologically generated by $\dot{X}, \dot{H}, \dot{Y}$. On the other hand, using (5.11) a thorough—but straightforward—computation shows that any element in $U_{\overline{h}}(\mathfrak{sl}_2)'$ does necessarily lie in U' (details are left to the reader; everything follows from definitions and the formulae for Δ^n). Thus $U_{\overline{h}}(\mathfrak{sl}_2)'$ is nothing but the unital subalgebra of $U_{\overline{h}}(\mathfrak{sl}_2)$ topologically generated by $\dot{X}, \dot{H}, \dot{Y}$. As a consequence, $U_{\overline{h}}(\mathfrak{sl}_2)'$ can be presented as the unital associative topological $\mathbb{K}[[\overline{h}]]$-algebra with (topological) generators $\dot{X}, \dot{H}, \dot{Y}$ and relations

$$\begin{aligned} &\dot{H}\dot{X} - \dot{X}\dot{H} = +2\overline{h}\dot{X}, \qquad \dot{H}\dot{Y} - \dot{Y}\dot{H} = -2\overline{h}\dot{Y}, \\ &\dot{X}\dot{Y} - \dot{Y}\dot{X} = \overline{h} A \cdot (\mathrm{e}^{+\dot{H}/2} - \mathrm{e}^{-\dot{H}/2}) = \overline{h} A \cdot (L^{+2} - L^{-2}), \end{aligned} \tag{5.12}$$

where $A := \overline{h}^2 / (\mathrm{e}^{+\overline{h}/2} - \mathrm{e}^{-\overline{h}/2}) = \overline{h} \cdot (\sum_{s>0} (+\overline{h}/2)^{2s} / (2s-1)!)^{-1} (\in \mathbb{K}[[\overline{h}]])$, with Hopf algebra structure given by

$$\begin{aligned} &\Delta(\dot{X}) = \dot{X} \otimes \mathrm{e}^{+\dot{H}/4} + \mathrm{e}^{-\dot{H}/4} \otimes \dot{X} = \dot{X} \otimes L^{+1} + L^{-1} \otimes \dot{X}, \\ &\Delta(\dot{H}) = \dot{H} \otimes 1 + 1 \otimes \dot{H}, \\ &\Delta(\dot{Y}) = \dot{Y} \otimes \mathrm{e}^{+\dot{H}/4} + \mathrm{e}^{-\dot{H}/4} \otimes \dot{Y} = \dot{Y} \otimes L^{+1} + L^{-1} \otimes \dot{Y}, \\ &\epsilon(\dot{X}) = \epsilon(\dot{H}) = \epsilon(\dot{Y}) = 0, \qquad \epsilon(L^{\pm 1}) = 1, \qquad S(\dot{X}) = -\mathrm{e}^{-\overline{h}/2} \dot{X} = -q^{-1} \dot{X}, \\ &S(\dot{H}) = -\dot{H}, \qquad (\dot{Y}) = -\mathrm{e}^{-\overline{h}/2} \dot{Y} = -q^{-1} \dot{Y}, \qquad S(L^{\pm 1}) = L^{\mp 1}. \end{aligned}$$

As an immediate consequence, this description yields also a similar presentation of $U_{\overline{h}}(\mathfrak{sl}_2)' / \overline{h} U_{\overline{h}}(\mathfrak{sl}_2)'$: then comparing the latter with the presentation of $F[[\mathfrak{sl}_2^*]]$ that one argues from Section 5.2 we find that, as predicted by the quantum duality principle (cf. Theorem 2.3) *there is an isomorphism of* (*topological*) *Poisson Hopf algebras*

$$\Phi_h : \frac{U_{\overline{h}}(\mathfrak{sl}_2)'}{\overline{h} U_{\overline{h}}(\mathfrak{sl}_2)'} = \mathbb{K}[[\dot{X}|_{\overline{h}=0}, \dot{H}|_{\overline{h}=0}, \dot{Y}|_{\overline{h}=0}]] \xrightarrow{\cong} F[[\mathfrak{sl}_2^*]] = k[[x, (z-1), y]],$$

where we set $S|_{\overline{h}=0} := S|\mathrm{mod}\, \overline{h} U_{\overline{h}}(\mathfrak{sl}_2)'$ for all $S \in U_{\overline{h}}(\mathfrak{sl}_2)'$ and the Poisson structure considered on $U_{\overline{h}}(\mathfrak{sl}_2)' / \overline{h} U_q^s(\mathfrak{sl}_2)'$ is the one given by the standard recipe (see Section 1.3 (b))

$$\{a|_{\overline{h}=0}, b|_{\overline{h}=0}\} := \left(\frac{ab - ba}{\overline{h}} \right) \Bigg|_{\overline{h}=0} \qquad \forall\, a, b \in U_{\overline{h}}(\mathfrak{sl}_2)'.$$

Explicitly, $\Phi_{\overline{h}}$ is given by

$$\dot{X}|_{\overline{h}=0} \mapsto x, \qquad \dot{H}|_{\overline{h}=0} \mapsto -4\,log(z), \qquad L^{\pm 1}|_{\overline{h}=0} \mapsto z^{\mp 1}, \qquad \dot{Y}|_{\overline{h}=0} \mapsto y. \tag{5.13}$$

Note also that the unital $\mathbb{K}[q, q^{-1}]$–subalgebra of $U_{\overline{h}}(\mathfrak{sl}_2)$—and of $U_q^s(\mathfrak{sl}_2)$—generated by $\check{X} := (q-1)X$, $L^{\pm 1}$, $\check{\Gamma} := (q-1)\Gamma$ and $\check{Y} := (q-1)Y$ is in fact a *Hopf* subalgebra, which we denote by $U_q^s(\mathfrak{sl}_2)'$ (note also that $\check{D} := (q-1)D = L - 1 \in U_q^s(\mathfrak{sl}_2)'$ too). Indeed,

$U_q^s(\mathfrak{sl}_2)'$ admits the presentation by the above generators and relations

$$\begin{aligned}
&L^{\pm1}L^{\mp1}=1,\qquad L^{\pm1}\check{\Gamma}=\check{\Gamma}L^{\pm1},\qquad (1+q^{-1})\check{\Gamma}=L^{+2}-L^{-2},\\
&\check{X}\check{Y}-\check{Y}\check{X}=(q-1)\check{\Gamma},\qquad L^{+2}-L^{-2}=(1+q^{-1})\check{\Gamma},\\
&L^{\pm1}\check{Y}=q^{\mp1}\check{Y}L^{\pm1},\qquad L^{\pm1}\check{X}=q^{\pm1}\check{X}L^{\pm1},\\
&\sqrt{\Gamma}\sqrt{Y}=q^{-2}\sqrt{Y}\sqrt{\Gamma}-(q-1)(q+q^{-1})\sqrt{F},\\
&\sqrt{\Gamma}\sqrt{X}=q^{+2}\sqrt{X}\sqrt{\Gamma}+(q-1)(q+q^{-1})\sqrt{X}
\end{aligned}$$

with Hopf structure given by

$$\begin{array}{lll}
\Delta(\sqrt{X})=\sqrt{X}\otimes L^{+1}+L^{-1}\otimes\sqrt{X} & \epsilon(\sqrt{X})=0 & S(\sqrt{X})=-q^{-1}\sqrt{X}\\
\Delta(\sqrt{\Gamma})=\sqrt{\Gamma}\otimes L^{+2}+L^{-2}\otimes\sqrt{\Gamma} & \epsilon(\sqrt{\Gamma})=0 & S(\sqrt{\Gamma})=-\sqrt{\Gamma}\\
\Delta(L^{\pm1})=L^{\pm1}\otimes L^{\pm1} & \epsilon(L^{\pm1})=1 & S(L^{\pm1})=L^{\mp1}\\
\Delta(\sqrt{Y})=\sqrt{Y}\otimes L^{+1}+L^{-1}\otimes\sqrt{Y} & \epsilon(\sqrt{Y})=0 & S(\sqrt{Y})=-q^{-1}\sqrt{Y}.
\end{array}$$

Similarly, the unital $\mathbb{K}[q,q^{-1}]$–subalgebra of $U_\hbar(\mathfrak{sl}_2)$—and of $U_q^a(\mathfrak{sl}_2)$ and $U_q^s(\mathfrak{sl}_2)'$—generated by $\sqrt{X}L^{-1}$, $K^{\pm1}$, $\sqrt{\Gamma}$ and $L^{+1}\sqrt{Y}$ is in fact a *Hopf* subalgebra too, which we denote by $U_q^a(\mathfrak{sl}_2)'$, and is of course a Hopf subalgebra of $U_q^s(\mathfrak{sl}_2)'$ as well (with $\sqrt{T}:=(q-1)T=K-1\in U_q^a(\mathfrak{sl}_2)'$ too).

Now this description yields also a similar presentation of $U_q^s(\mathfrak{sl}_2)'/(q-1)U_q^s(\mathfrak{sl}_2)'$: then comparing the latter with the presentation of $F[{}_s\mathrm{SL}_2^*]$ in Section 5.2 we find that *there is a Poisson Hopf algebra isomorphism*

$$\Phi_q^s:\frac{U_q^s(\mathfrak{sl}_2)'}{(q-1)U_q^s(\mathfrak{sl}_2)'}\xrightarrow{\cong}F[{}_s\mathrm{SL}_2^*],$$

where we set $S|_{q=1}:=S\bmod(q-1)U_q^s(\mathfrak{sl}_2)'$ for all $S\in U_q^s(\mathfrak{sl}_2)'$ and the Poisson structure considered on $U_q^s(\mathfrak{sl}_2)'/(q-1)U_q^s(\mathfrak{sl}_2)$ is the one given by the standard recipe

$$\{a|_{q=1},b|_{q=1}\}:=\left.\left(\frac{ab-ba}{(q-1)}\right)\right|_{q=1}\qquad\forall\,a,b\in U_q^s(\mathfrak{sl}_2)'.$$

Explicitly, Φ_q^s is given by

$$\begin{aligned}
&\sqrt{X}|_{q=1}\mapsto\tfrac{1}{2}x,\qquad \sqrt{\Gamma}|_{q=1}\mapsto\tfrac{1}{2}(z^{-2}-z^{+2}),\\
&L^{\pm1}|_{q=1}\mapsto z^{\mp1},\qquad \sqrt{Y}|_{q=1}\mapsto\tfrac{1}{2}y.
\end{aligned}$$

In addition, Φ_q^s gives by restriction a similar Poisson Hopf algebra isomorphism

$$\Phi_q^a:\frac{U_q^a(\mathfrak{sl}_2)'}{(q-1)U_q^a(\mathfrak{sl}_2)'}\xrightarrow{\cong}F[{}_a\mathrm{SL}_2^*],\qquad (\sqrt{X}L^{-1})|_{q=1}\mapsto\frac{xz}{2},$$

$$\sqrt{\Gamma}|_{q=1}\mapsto\frac{z^{-2}-z^{+2}}{2},\qquad K^{\pm1}|_{q=1}\mapsto z^{\mp2},\qquad (L^{+1}\sqrt{Y})|_{q=1}\mapsto\frac{z^{-1}y}{2}.$$

The reason for considering $U_q^c(\mathfrak{sl}_2)$ and $U_q^c(\mathfrak{sl}_2)'$ (for $c = a, s$) is that we can compute the braiding $\mathfrak{R}_{\mathrm{GH}}$ through them, as we shall see in the sequel.

First, $U_{\bar{h}}(\mathfrak{sl}_2)$ is indeed a $\mathcal{QTQUEA}$, whose R-matrix is $R_{\bar{h}} = R_0 \cdot R_1$ with

$$R_0 = \exp\left(\bar{h} \cdot \frac{H \otimes H}{4}\right),$$

$$R_1 = \sum_{n \in \mathbb{N}} \frac{(\mathrm{e}^{\bar{h}})^{\binom{n+1}{2}}}{(n)_{\mathrm{e}^{\bar{h}}}!} (\mathrm{e}^{\bar{h}} - 1)^n \cdot (\mathrm{e}^{+\bar{h}H/4} X)^n \otimes (\mathrm{e}^{-\bar{h}H4} Y)^n,$$

where $(n)_a! := \prod_{r=1}^{n} (a^n - 1)/(a - 1)$ (in this case $a = \mathrm{e}^{\bar{h}}$). This R-matrix is a quantization of the classical $\boldsymbol{r}$-matrix of $\mathfrak{sl}_2$, in the sense that $R_{\bar{h}} = 1 + \boldsymbol{r}\bar{h} + \mathcal{O}(\bar{h}^2)$, where $\mathcal{O}(\bar{h}^2)$ is some element of $\bar{h}^2 \cdot U_{\bar{h}}(\mathfrak{g}) \otimes U_{\bar{h}}(\mathfrak{g})$ (like in Remark 1.9 (b)); thus the $\mathcal{QTQUEA}$ $(U_{\bar{h}}(\mathfrak{sl}_2), R_{\bar{h}})$ is a quantization of the quasitriangular Lie bialgebra $(\mathfrak{sl}_2, \boldsymbol{r})$, as required to ignite the quantization deformation procedure to construct a braiding on $F[[\mathfrak{g}^* \oplus \mathfrak{g}^*]]$ for $\mathfrak{g} = \mathfrak{sl}_2$.

Now, we are interested in the braiding operator $\mathfrak{R}_{\mathrm{GH}}$ induced at $\bar{h} = 0$ by the operator $\mathfrak{R}_{\bar{h}} := \mathrm{Ad}(R_{\bar{h}})$ acting on the algebra $(U_{\bar{h}}(\mathfrak{sl}_2) \hat{\otimes} U_{\bar{h}}(\mathfrak{sl}_2))' = U_{\bar{h}}(\mathfrak{sl}_2)' \tilde{\otimes} U_{\bar{h}}(\mathfrak{sl}_2)'$.

We perform the calculation along the following lines. As the R-matrix factors into $R_{\bar{h}} = R_0 \cdot R_1$, we compute separately the adjoint action of the two factors onto $U_{\bar{h}}(\mathfrak{sl}_2)' \tilde{\otimes} U_{\bar{h}}(\mathfrak{sl}_2)'$ modulo $\bar{h}$. A first analysis shows that both actions are given by exponentials of Hamiltonian vector fields on the formal Poisson group $\mathfrak{sl}_2^* \times \mathfrak{sl}_2^*$. The first action—namely, that arising from R_0—is computed via straightforward calculation. As for the second action—the one of R_1—one in fact has to compute the action of a Hamiltonian vector field on ${}_s\mathrm{SL}_2^* \times {}_s\mathrm{SL}_2^*$ (minus a divisor): using Leibniz' rule, one reduces to compute the action of some Hamiltonian vector fields on ${}_s\mathrm{SL}_2^*$ alone.

To begin with, write $R_{\bar{h}} = R_0 \cdot R_1$ in terms of $U_{\bar{h}}(\mathfrak{sl}_2)'$: like in [8, Section 4], we find

$$R_0 = \exp\left(\bar{h} \cdot H \otimes \frac{H}{4}\right) = \exp\left(\bar{h}^{-1} \cdot \dot{H} \otimes \frac{\dot{H}}{4}\right),$$

$$\begin{aligned} R_1 &= \sum_{n \in \mathbb{N}} \frac{(\mathrm{e}^{\bar{h}})^{\binom{n+1}{2}}}{(n)_{\mathrm{e}^{\bar{h}}!}} (\mathrm{e}^{\bar{h}} 1)^n \cdot (\mathrm{e}^{+\bar{h}H/4} X)^n \otimes (\mathrm{e}^{-\bar{h}H/4} Y)^n \\ &= ((\mathrm{e}^{\bar{h}} - 1)^2 \cdot L^{+1} X \otimes L^{-1} Y; \mathrm{e}^{\bar{h}})_\infty, \end{aligned}$$

where $(z; q)_\infty := \prod_{n \in \mathbb{N}} (1 - zq^n)$. Now, the behavior of R_1 when $\bar{h} \to 0$ is ruled by [14], Lemma 3.4.1 (see also [8], Lemma 2.2), namely (proceeding as in [9, Section 4]), we have

$$\begin{aligned} R_1 &= \exp\left(\frac{-1}{\bar{h}} \cdot \int_0^{(\mathrm{e}^{\bar{h}} - 1)^2 \cdot L^{+1} X \otimes L^{-1} Y} \frac{\log(1 - \tau)}{\tau} \,\mathrm{d}\tau \cdot (1 + \bar{h}C)\right) \\ &= \exp\left(\frac{-1}{\bar{h}} \cdot \int_0^{L^{+1} \dot{X} \otimes L^{-1} \dot{Y}} \frac{\log(1 + t)}{t,} \,\mathrm{d}t \cdot (1 + \bar{h}K)\right), \end{aligned}$$

where $\int_0^{L^{+1}\dot{X}\otimes L^{-1}\dot{Y}}(\log(1+t)/t)\,\mathrm{d}t := \sum_{n>0}(L^{+1}\dot{X}\otimes L^{-1}\dot{Y})/n^2$ (use Mac Laurin expansion of $\log(1+x)$) and C and K denote suitable elements of $U_{\overline{h}}(\mathfrak{sl}_2)'\tilde{\otimes}U_{\overline{h}}(\mathfrak{sl}_2)'$, namely again power series in $L^{+1}\dot{X}\otimes L^{-1}\dot{Y}$, hence they commute with $\int_0^{L^{+1}\dot{X}\otimes L^{-1}\dot{Y}}(\log(1-t)/t)\,\mathrm{d}t$; so

$$R_1 = \exp\left(\frac{-1}{\overline{h}}\cdot\int_0^{L^{+1}\dot{X}\otimes L^{-1}\dot{Y}}\frac{\log(1+t)}{t}\,\mathrm{d}t\cdot(1+\overline{h}K)\right) = \exp\left(\frac{-1}{\overline{h}}\cdot\int_0^{L^{+1}\dot{X}\otimes L^{-1}\dot{Y}}\frac{\log(1+t)}{t}\,\mathrm{d}t\right)\cdot Z$$

for some $Z\in U_{\overline{h}}(\mathfrak{sl}_2)'\tilde{\otimes}U_{\overline{h}}(\mathfrak{sl}_2)'$. Of course we have

$$\mathfrak{R}_{\overline{h}} := \mathrm{Ad}(R_{\overline{h}}) = \mathrm{Ad}(R_0\cdot R_1) = \mathrm{Ad}(R_0)\circ\mathrm{Ad}(R_1) = \mathfrak{R}_{\overline{h}}^{(0)}\circ\mathfrak{R}_{\overline{h}}^{(1)}$$

with $\mathfrak{R}_{\overline{h}}^{(0)} := \mathrm{Ad}(R_0)$, $\mathfrak{R}_{\overline{h}}^{(1)} := \mathrm{Ad}(R_1)$. Thus also

$$\mathfrak{R}_{\mathrm{GH}} := \mathfrak{R}_{\overline{h}}|_{\overline{h}=0} = \mathfrak{R}_{\mathrm{GH}}^{(0)}\circ\mathfrak{R}_{\mathrm{GH}}^{(1)} \quad \text{with}\quad \mathfrak{R}_{\mathrm{GH}}^{(0)} := \mathfrak{R}_{\overline{h}}^{(0)}|_{\overline{h}=0},\qquad \mathfrak{R}_{\mathrm{GH}}^{(1)} := \mathfrak{R}_{\overline{h}}^{(1)}|_{\overline{h}=0}. \tag{5.14}$$

Finally we have

$$\begin{aligned}\mathfrak{R}_{\overline{h}}^{(1)} &:= \mathrm{Ad}(R_1) = \mathrm{Ad}\left(\exp\left(\frac{-1}{\overline{h}}\cdot\int_0^{L^{+1}\dot{X}\otimes L^{-1}\dot{Y}}\frac{\log(1+t)}{t}\,\mathrm{d}t\right)\cdot Z\right)\\ &= \mathrm{Ad}\left(\exp\left(\frac{-1}{\overline{h}}\cdot\int_0^{L^{+1}\dot{X}\otimes L^{-1}\dot{Y}}\frac{\log(1+t)}{t}\,\mathrm{d}t\right)\right)\circ\mathrm{Ad}(Z)\\ &= \mathrm{Ad}\left(\exp\left(\frac{-1}{\overline{h}}\cdot\int_0^{L^{+1}\dot{X}\otimes L^{-1}\dot{Y}}\frac{\log(1+t)}{t}\,\mathrm{d}t\right)\right) \mod \overline{h}\cdot U_{\overline{h}}(\mathfrak{sl}_2)'\otimes U_{\overline{h}}(\mathfrak{sl}_2)'\end{aligned}$$

because $Z\in U_{\overline{h}}(\mathfrak{sl}_2)'\tilde{\otimes}U_{\overline{h}}(\mathfrak{sl}_2)'$ and $(U_{\overline{h}}(\mathfrak{sl}_2)'\tilde{\otimes}U_{\overline{h}}(\mathfrak{sl}_2)')|_{\overline{h}=0} = F[[\mathfrak{sl}_2^*\times\mathfrak{sl}_2^*]]$ is commutative (hereafter, by $S|_{\overline{h}=0}$ we shall denote the coset of $S\in U_{\overline{h}}(\mathfrak{sl}_2)'\tilde{\otimes}U_{\overline{h}}(\mathfrak{sl}_2)'$ modulo $\overline{h}\cdot U_{\overline{h}}(\mathfrak{sl}_2)'\tilde{\otimes}U_{\overline{h}}(\mathfrak{sl}_2)'$). Hence our analysis shows that $\mathfrak{R}_{\overline{h}}^{(i)} = \mathrm{Ad}(\exp(\overline{h}^{-1}\Lambda_i))$ with $\Lambda_i\in U_{\overline{h}}(\mathfrak{sl}_2)'\tilde{\otimes}U_{\overline{h}}(\mathfrak{sl}_2)'$ for $i=0,1$. Indeed, we found

$$\Lambda_0 = \frac{(\dot{H}\otimes\dot{H})}{4},\quad \Lambda_1 = -\int_0^{L^{+1}\dot{X}\otimes L^{-1}\dot{Y}}\frac{\log(1+t)}{t}\,\mathrm{d}t = -\sum_{n>0}\frac{(L^{+1}\dot{X}\otimes L^{-1}\dot{Y})^n}{n^2}.$$

But then we have $\mathfrak{R}_{\overline{h}}^{(i)} = \mathrm{Ad}(\exp(\overline{h}^{-1}\Lambda_i)) = \exp(\mathrm{ad}_{[,]}(\overline{h}^{-1}\Lambda_i)) = \exp(\mathrm{ad}_{\frac{[,]}{\overline{h}}}(\Lambda_i)) \equiv \exp(\mathrm{ad}_{\{,\}}(\Lambda_i|_{\overline{h}=0}))$, that is $\mathfrak{R}_{\mathrm{GH}}^{(i)} = \exp(\mathrm{ad}_{\{,\}}(\Lambda_i|_{\overline{h}=0}))$. In geometric terms, this means that $\mathfrak{R}_{\mathrm{GH}}^{(i)}$ (hence also $\mathfrak{R}_{\mathrm{GH}}$) is the integration of a Hamiltonian vector fields over the formal Poisson group $\mathfrak{sl}_2^*\times\mathfrak{sl}_2^*$.

To describe $\mathfrak{R}_{\mathrm{GH}}^{(0)}$ and $\mathfrak{R}_{\mathrm{GH}}^{(1)}$ we set $S_1 := S\otimes 1$, $S_2 := 1\otimes S$ for any $S\in U_{\overline{h}}(\mathfrak{sl}_2)$ (note that S_1 and S_2 commute with each other) and also $\bar{S}$ for any coset modulo $\overline{h}$.

The case of $\mathfrak{R}_{\mathrm{GH}}^{(0)}$ is trivial: direct computation—using (5.12)—gives

$$\begin{aligned}
&\mathfrak{R}_{\hbar}^{(0)}(\dot{X}_1) = \dot{X}_1 L_2^{+2}, \qquad \mathfrak{R}_{\hbar}^{(0)}(\dot{H}_1) = \dot{H}_1, \qquad \mathfrak{R}_{\hbar}^{(0)}(\dot{L}_1^{\pm 1}) = \dot{L}_1^{\pm 1},\\
&\mathfrak{R}_{\hbar}^{(0)}(\dot{Y}_1) = \dot{Y}_1 L_2^{-2}, \qquad \mathfrak{R}_{\hbar}^{(0)}(\dot{X}_2) = L_1^{+2}\dot{X}_2, \qquad \mathfrak{R}_{\hbar}^{(0)}(\dot{H}_2) = \dot{H}_2,\\
&\mathfrak{R}_{\hbar}^{(0)}(\dot{L}_2^{\pm 1}) = \dot{L}_2^{\pm 1}, \qquad \mathfrak{R}_{\hbar}^{(0)}(\dot{Y}_2) = L_1^{-2}\dot{Y}_2,
\end{aligned}$$

whence using (5.13) we argue at once for $\mathfrak{R}_{\mathrm{GH}}^{(0)} : F[[\mathfrak{sl}_2^* \oplus \mathfrak{sl}_2^*]] \stackrel{\cong}{\rightarrow} F[[\mathfrak{sl}_2^* \oplus \mathfrak{sl}_2^*]]$

$$\begin{aligned}
&\mathfrak{R}_{\mathrm{GH}}^{(0)}(x_1) = x_1 z_2^{-2}, \qquad \mathfrak{R}_{\mathrm{GH}}^{(0)}(z_1^{\pm 1}) = z_1^{\pm 1}, \qquad \mathfrak{R}_{\mathrm{GH}}^{(0)}(y_1) = y_1 z_2^{+2},\\
&\mathfrak{R}_{\mathrm{GH}}^{(0)}(x_2) = z_1^{-2} x_2, \qquad \mathfrak{R}_{\mathrm{GH}}^{(0)}(z_2^{\pm 1}) = z_2^{\pm 1}, \qquad \mathfrak{R}_{\mathrm{GH}}^{(0)}(y_2) = z_1^{+2} y_2
\end{aligned} \tag{5.15}$$

(recall that $F[[\mathfrak{sl}_2^* \oplus \mathfrak{sl}_2^*]] = \mathbb{K}[[x_1, (z_1 - 1), y_1, x_2, (z_2 - 1), y_2]]$ thus $\mathfrak{R}_{\mathrm{GH}}^{(0)}$ is uniquely determined by the images of x_i, z_i, y_i $[i = 1, 2]$).

As for $\mathfrak{R}_{\mathrm{GH}}^{(1)}$, we proceed in steps. First, using the Jacobi identity for $\{,\}$ we get

$$\begin{aligned}
\mathfrak{R}_{\mathrm{GH}}^{(1)} &= \exp\left(\mathrm{ad}_{\{,\}}\left(-\int_0^{L_1^{+1}\dot{X}_1 L_2^{-1}\dot{Y}_2} \frac{\log(1+t)}{t}\,\mathrm{d}t\bigg|_{\hbar=0}\right)\right)\\
&= \exp\left(\mathrm{ad}_{\{,\}}\left(-\int_0^{z_1^{-1}x_1 z_2^{+1} y_2} \frac{\log(1+t)}{t}\,\mathrm{d}t\right)\right)\\
&= \exp\left(\mu\left(-\frac{\log(1+z_1^{-1}x_1 z_2^{+1} y_2)}{z_1^{-1}x_1 z_2^{+1} y_2}\right) \circ \mathrm{ad}_{\{,\}}(z_1^{-1}x_1 z_2^{+1} y_2)\right),
\end{aligned}$$

where $\mu(S)$ denotes the operator of left multiplication by $S \in F[[\mathfrak{sl}_2^* \oplus \mathfrak{sl}_2^*]]$. Indeed

$$\begin{aligned}
&\mathrm{ad}_{\{,\}}\left(-\int_0^{z_1^{-1}x_1 z_2^{+1} y_2} \frac{\log(1+t)}{t}\,\mathrm{d}t\right)(\ell) = \mathrm{ad}_{\{,\}}\left(-\sum_{n>0} \frac{(z_1^{-1}x_1 z_2^{+1} y_2)}{n^2}\right)(\ell)\\
&= -\sum_{n>0} \frac{1}{n^2} \cdot \{(z_1^{-1}x_1 z_2^{+1} y_2)^n, \ell\} = -\sum_{n>0} \frac{1}{n^2} n (z_1^{-1}x_1 z_2^{+1} y_2)^{n-1} \cdot \{z_1^{-1}x_1 z_2^{+1} y_2, \ell\}\\
&= -\sum_{n>0} \frac{(z_1^{-1}x_1 z_2^{+1} y_2)^{n-1}}{n} \cdot \{z_1^{-1}x_1 z_2^{+1} y_2, \ell\}\\
&= -\frac{\log(1+z_1^{-1}x_1 z_2^{+1} y_2)}{z_1^{-1}x_1 z_2^{+1} y_2} \cdot \{z_1^{-1}x_1 z_2^{+1} y_2, \ell\}
\end{aligned}$$

(because of Jacobi identity: $\{\cdot, \ell\} = -\mathrm{ad}_{\{,\}}(\ell)$ is a derivation!). Secondly, again by the Jacobi identity and the commutation relation $z_1^{-1}x_1 \cdot z_2^{+1} y_2 = z_2^{+1} y_2 \cdot z_1^{-1} x_1$ we get

$$\mu\left(-\frac{\log(1+z_1^{-1}x_1z_2^{+1}y_2)}{z_1^{-1}x_1z_2^{+1}y_2}\right)\circ \mathrm{ad}_{\{,\}}(z_1^{-1}x_1z_2^{+1}y_2)$$
$$=\mu\left(-\frac{\log(1+z_1^{-1}x_1z_2^{+1}y_2)}{z_1^{-1}x_1z_2^{+1}y_2}\right)\circ \mathrm{ad}_{\{,\}}(z_1^{-1}x_1)\circ\mu(z_2^{+1}y_2)$$
$$+\mu\left(-\frac{\log(1+z_1^{-1}x_1z_2^{+1}y_2)}{z_1^{-1}x_1z_2^{+1}y_2}\right)\circ\mu(z_1^{-1}x_1)\circ \mathrm{ad}_{\{,\}}(z_2^{+1}y_2)$$
$$=\mu\left(-\frac{\log(1+z_1^{-1}x_1z_2^{+1}y_2)}{z_1^{-1}x_1}\right)\circ \mathrm{ad}_{\{,\}}(z_1^{-1}x_1)$$
$$+\mu\left(-\frac{\log(1+z_2^{+1}y_2)}{z_1^{-1}x_1z_2^{+1}y_2}\right)\circ \mathrm{ad}_{\{,\}}(z_2^{+1}y_2).$$

The two summands above are mutually commuting operators—thanks to the commutation relation $z_1^{-1}x_1\cdot z_2^{+1}y_2=z_2^{+1}y_2\cdot z_1^{-1}x_1$—so when we take the exponential we get

$$\mathfrak{R}_{\mathrm{GH}}^{(1)}=\exp\left(\mu\left(-\frac{\log(1+z_1^{-1}x_1z_2^{+1}y_2)}{z_1^{-1}x_1z_2^{+1}y_2}\right)\circ \mathrm{ad}_{\{,\}}(z_1^{-1}x_1z_2^{+1}y_2)\right)$$
$$=\exp\left(\mu\left(-\frac{\log(1+z_1^{-1}x_1z_2^{+1}y_2)}{z_1^{-1}x_1}\right)\circ \mathrm{ad}_{\{,\}}(z_1^{-1}x_1)\right.$$
$$\left.+\mu\left(-\frac{\log(1+z_1^{-1}x_1z_2^{+1}y_2)}{z_2^{+1}y_2}\right)\circ \mathrm{ad}_{\{,\}}(z_2^{+1}y_2)\right)$$
$$=\exp\left(\mu\left(-\frac{\log(1+z_1^{-1}x_1z_2^{+1}y_2)}{z_1^{-1}x_1}\right)\circ \mathrm{ad}_{\{,\}}(z_1^{-1}x_1)\right)\circ$$
$$\exp\left(\mu\left(-\frac{\log(1+z_1^{-1}x_1z_2^{+1}y_2)}{z_2^{+1}y_2}\right)\circ \mathrm{ad}_{\{,\}}(z_2^{+1}y_2)\right).$$

In a nutshell, we have

$$\mathfrak{R}_{\mathrm{GH}}^{(1)}=\exp(\mathcal{E}_1)\circ\exp(\mathcal{F}_2)\quad\text{with}\quad\begin{cases}\mathcal{E}_1:=\mu\left(-\dfrac{\log(\nabla^2)}{z_1^{-1}x_1}\right)\circ \mathrm{ad}_{\{,\}}(z_1^{-1}x_1),\\[2ex] \mathcal{F}_2:=\mu\left(-\dfrac{\log(\nabla^2)}{z_2^{+1}y_2}\right)\circ \mathrm{ad}_{\{,\}}(z_2^{+1}y_2),\end{cases}\tag{5.16}$$

where $\nabla:=(1+z_1^{-1}x_1z_2^{+1}y_2)^{1/2}$. We proceed now with computations.

Since $\{s_1,r_2\}=0$ for all $s,r\in F[[\mathfrak{sl}_2^*]]$, we have $\mathcal{E}_1(r_2)=0$ for all $r\in F[[\mathfrak{sl}_2^*]]$, so

$$\exp(\mathcal{E}_1)(x_2)=x_2,\qquad \exp(\mathcal{E}_1)(z_2^{\pm1})=z_2^{\pm1},\qquad \exp(\mathcal{E}_1)(y_2)=y_2.\tag{5.17}$$

Now for the rest! We have to compute things like $\{s_1,r_1\}$, so for simplicity we shall drop the subscript 1 throughout.

For the operator $\mathrm{ad}_{\{,\}}(z^{-1}x)$ (a derivation!) we have the formulæ

$$\{z^{-1}x, x\} = \{z^{-1}, x\} \cdot x + z^{-1} \cdot \{x, x\} = \{z^{-1}, x\} \cdot x = \left(z^{-1}\tfrac{1}{2}x\right) \cdot x,$$
$$\{z^{-1}x, z^{\pm 1}\} = \{z^{-1}, z^{\pm 1}\} \cdot x + z^{-1} \cdot \{x, z^{\pm 1}\} = z^{-1} \cdot \{x, z^{\pm 1}\} = \pm\left(z^{-1}\tfrac{1}{2}x\right) \cdot z^{\pm 1},$$
$$\{z^{-1}x, y\} = \{z^{-1}, y\} \cdot x + z^{-1} \cdot \{x, y\} = -\left(\tfrac{1}{2}z^{-1}\right) \cdot y \cdot x + z^{-1} \cdot (z^{-2} - z^{+2})$$
$$= -\left(z^{-1}\tfrac{1}{2}x\right) \cdot y + z^{-3} - z^{+1},$$
$$\{z^{-1}x, z^{+1}y\} = \{z^{-1}x, z^{+1}\} \cdot y + z^{+1} \cdot \{z^{-1}x, y\} = z^{-2} - z^{+2}.$$

Then for $\exp(\mathcal{E}_1) = \exp(\mu(-\log(\nabla^2)/z_1^{-1}x_1) \circ \mathrm{ad}_{\{,\}}(z_1^{-1}x_1))$ we have

$$\exp(\mathcal{E}_1)(x_1) = \exp\left(-\tfrac{1}{2}\log(\nabla^2)\right) \cdot x_1 = \exp(-\log(\nabla)) \cdot x_1 = x_1 \cdot \nabla^{-1},$$
$$\exp(\mathcal{E}_1)(z_1^{\pm 1}) = \exp\left(\mp\tfrac{1}{2}\log(\nabla^2)\right) \cdot z_1^{\pm 1} = \exp(\mp\log(\nabla)) \cdot z_1^{\pm 1} = z_1^{\pm 1} \cdot \nabla^{\mp 1},$$
$$\exp(\mathcal{E}_1)(y_1) = y_1 \cdot \nabla^{+1} + y_2 \cdot z_2^{+1}z_1^{-3} \cdot \nabla^{+1} - y_2 \cdot z_2^{+1}z_1^{+1} \cdot \nabla^{-1}, \tag{5.18}$$

where the latter identity is computed (since $\exp(\mathcal{E}_1)$ is an automorphism!) as follows:

$$\begin{aligned}
\exp(\mathcal{E}_1)(y_1) &= \exp(\mathcal{E}_1)(z_1^{-1} \cdot z_1^{+1}y_1) = \exp(\mathcal{E}_1)(z_1^{-1}) \cdot \exp(\mathcal{E}_1(z_1^{+1}y_1)) \\
&= z_1^{-1} \cdot \nabla^{+1} \cdot \left(z_1^{+1}y_1 + \sum_{n>1}\frac{1}{n!} \cdot \left(-\frac{\log(\nabla^2)}{z_1^{-1}x_1}\right)^n \cdot \mathcal{E}_1^{n-1}(z_1^{-2} - z_1^{+2})\right) \\
&= z_1^{-1} \cdot \nabla^{+1} \cdot \left(z_1^{+1}y_1 + \sum_{n>0}\frac{1}{n!} \cdot \left(-\frac{\log(\nabla^2)}{z_1^{-1}x_1}\right)^n \cdot (-z_1^{-1}x_1)^{n-1} \cdot z_1^{-2}\right. \\
&\qquad \left. - \sum_{n>0}\frac{1}{n!} \cdot \left(-\frac{\log(\nabla^2)}{z_1^{-1}x_1}\right)^n \cdot (+z_1^{-1}x_1)^{n-1} \cdot z_1^{+2}\right) \\
&= z_1^{-1} \cdot \nabla^{+1} \cdot \left(z_1^{+1}y_1 + \frac{\exp(+\log(\nabla^2)) - 1}{+z_1^{-1}x_1} \cdot z_1^{-2}\right. \\
&\qquad \left. - \frac{\exp(-\log(\nabla^2)) - 1}{-z_1^{-1}x_1} \cdot z_1^{+2}\right) \\
&= z_1^{-1} \cdot \nabla^{+1} \cdot \left(z_1^{+1}y_1 + \frac{\nabla^{+2} - 1}{+z_1^{-1}x_1} \cdot z_1^{-2} - \frac{\nabla^{-2} - 1}{-z_1^{-1}x_1} \cdot z_1^{+2}\right) \\
&= z_1^{-1} \cdot \nabla^{+1} \cdot (z_1^{+1}y_1 + y_2 \cdot z_2^{+1}z_1^{-2} - y_2 \cdot z_2^{+1}z_1^{+2} \cdot \nabla^{-2}) \\
&= y_1 \cdot \nabla^{+1} + y_2 \cdot z_2^{+1}z_1^{-3} \cdot \nabla^{+1} - y_2 \cdot z_2^{+1}z_1^{+1} \cdot \nabla^{-1}.
\end{aligned}$$

Now for $\exp(\mathcal{F}_2)$. Again, since $\{s_1, r_2\} = 0$ for all $s, r \in F[[\mathfrak{sl}_2^*]]$ we have $\mathcal{F}_2(s_1) = 0$ for all $s \in F[[\mathfrak{sl}_2^*]]$, so

$$\exp(\mathcal{F}_2)(x_1) = x_1, \qquad \exp(\mathcal{F}_2)(z_1^{\pm 1}) = z_1^{\pm 1}, \qquad \exp(\mathcal{F}_2)(y_1) = y_1. \tag{5.19}$$

As for the rest, we can base upon the previous results, as follows. First, we note that there is a *Poisson algebra* automorphism

$$\Phi : F[[\mathfrak{sl}_2^*]] \xrightarrow{\cong} F[[\mathfrak{sl}_2^*]], \qquad x \mapsto y, \qquad z^{\pm 1} \mapsto z^{\mp 1}, \qquad y \mapsto x$$

such that $\Phi^{-1} = \Phi$ (and which also restrict to $F[{}_s\mathrm{SL}_2^*]$ and to $F[{}_a\mathrm{SL}_2^*]$). Then we have immediately from definitions that $(\Phi \otimes \Phi)(\mathcal{E}_1)(s \otimes r) = \sigma(\mathcal{F}_2(\Phi(r) \otimes \Phi(s))$ for all $s, r \in F[[\mathfrak{sl}_2^*]]$ (with σ as in Definition 2.4), whence in particular we argue

$$\mathcal{F}_2(s_2) = \sigma(\Phi^{\otimes 2}(\mathcal{E}_1(\Phi^{-1}(s_1)))) = \sigma(\Phi^{\otimes 2}(\mathcal{E}_1(\Phi(s_1)))) \quad \forall s \in F[[\mathfrak{sl}_2^*]]$$

and so

$$\exp(\mathcal{F}_2)(s_2) = \sigma(\Phi^{\otimes 2}(\exp(\mathcal{E}_1)(\Phi(s_1)))) \quad \forall s \in F[[\mathfrak{sl}_2^*]].$$

Using this and formulæ (5.18) we eventually get

$$\begin{aligned} &\exp(\mathcal{F}_2)(x_2) = x_2 \cdot \nabla^{+1} + x_1 \cdot z_1^{-1} z_2^{+3} \cdot \nabla^{+1} - x_1 \cdot z_1^{-1} z_2^{-1} \cdot \nabla^{-1}, \\ &\exp(\mathcal{F}_2)(z_2^{\pm 1}) = z_2^{\pm 1} \cdot \nabla^{\pm 1}, \qquad \exp(\mathcal{F}_2)(y_1) = y_2 \cdot \nabla^{-1}. \end{aligned} \tag{5.20}$$

Formulæ (5.16)–(5.20) give us a complete description of $\mathfrak{R}_{\mathrm{GH}}^{(1)}$. To summarize, it is given

$$\begin{aligned} &\mathfrak{R}_{\mathrm{GH}}^{(1)}(x_1) = x_1 \cdot \nabla^{-1}, \qquad \mathfrak{R}_{\mathrm{GH}}^{(1)}(z_1^{\pm 1}) = z_1^{\pm 1} \cdot \nabla^{\mp 1} \\ &\mathfrak{R}_{\mathrm{GH}}^{(1)}(y_1) = y_1 \cdot \nabla^{+1} + y_2 \cdot z_2^{+1} z_1^{-3} \cdot \nabla^{+1} - y_2 \cdot z_2^{+1} z_1^{+1} \cdot \nabla^{-1}, \\ &\mathfrak{R}_{\mathrm{GH}}^{(1)}(x_2) = x_2 \cdot \nabla^{+1} + x_1 \cdot z_1^{-1} z_2^{+3} \cdot \nabla^{+1} - x_1 \cdot z_1^{-1} z_2^{-1} \cdot \nabla^{-1}, \\ &\mathfrak{R}_{\mathrm{GH}}^{(1)}(z_2^{\pm 1}) = z_2^{\pm 1} \cdot \nabla^{\pm 1}, \qquad \mathfrak{R}_{\mathrm{GH}}^{(1)}(y_2) = y_2 \cdot \Delta^{-1}. \end{aligned} \tag{5.21}$$

Finally, composing with $\mathfrak{R}_{\mathrm{GH}}^{(0)}$—see (5.15)—we find at last

$$\begin{aligned} &\mathfrak{R}_{\mathrm{GH}}(x_1) = x_1 \cdot z_2^{-2} \cdot \Theta^{-1}, \qquad \mathfrak{R}_{\mathrm{GH}}(z_1^{\pm 1}) = z_1^{\pm 1} \cdot \Theta^{\mp 1}, \\ &\mathfrak{R}_{\mathrm{GH}}(y_1) = y_1 \cdot z_2^{+2} \cdot \Theta^{+1} + y_2 \cdot z_2^{+1} z_1^{-1} \cdot \Theta^{+1} - y_2 \cdot z_2^{+1} z_1^{+3} \cdot \Theta^{-1}, \\ &\mathfrak{R}_{\mathrm{GH}}(x_2) = x_2 \cdot z_1^{-2} \cdot \Theta^{+1} + x_1 \cdot z_1^{-1} z_2^{+1} \cdot \Theta^{+1} - x_1 \cdot z_1^{-1} z_2^{-3} \cdot \Theta^{-1}, \\ &\mathfrak{R}_{\mathrm{GH}}(z_2^{\pm 1}) = z_2^{\pm 1} \cdot \Theta^{\pm 1}, \qquad \mathfrak{R}_{\mathrm{GH}}(y_2) = y_2 \cdot z_1^{+2} \cdot \Theta^{-1} \end{aligned} \tag{5.22}$$

for $\mathfrak{R}_{\mathrm{GH}} = \mathfrak{R}_{\mathrm{GH}}^{(0)} \circ \mathfrak{R}_{\mathrm{GH}}^{(1)}$ (see (5.14)), with $\Theta := (1 + x_1 z_1^{+1} z_2^{-1} y_2)^{1/2} = \mathfrak{R}_{\mathrm{GH}}^{(0)}(\nabla)$.

Therefore, just comparing (5.22) with (5.8) we get as an outcome the main result of this section.

Theorem 5.2. *The braidings $\mathfrak{R}_{\mathrm{WX}}$ and $\mathfrak{R}_{\mathrm{GH}}$ for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ do coincide. In other words, the answer to the "question" in Section 5.1 is positive for $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$.*

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